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An Investigation of Stochastic Cooling in the Framework of Control Theory

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Systemtheoretische Untersuchung der stochastischen Kühlung

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Kurzfassung

Diese Arbeit betrachtet das stochastische Kühlen ungebunchter Strahlen unter dem Aspekt der Systemtheorie. Der Schwerpunkt liegt dabei auf der Untersuchung der kollektiven Strahlbewegung, mit dem Ziel, Stabilitätsaussagen für einen Strahl in einem aktiven Kühlsystem zu erhalten. Denn ein stochastisches Kühlsystem bildet einen Rückkopplungskreis und ist daher vergleichbar mit den Feedback-Systemen, die zur Dämpfung kollektiver Instabilitäten eingesetzt werden. Da jedes System, das auf sich selbst zurückwirkt, potentiell instabil ist, erfordern derartige Rückkopplungskreise eine sorgfältige Analyse ihrer Stabilität.

Ausgehend von einer linearen Kühlwechselwirkung wird für das transversale Kühlen eine selbstkonsistente Lösung der Strahlbewegung hergeleitet. Dazu wird die kollektive Bewegung des Strahls in seine kohärenten Moden zerlegt. Die Rechnung berücksichtigt die Lokalität von Detektor und Kicker und die daraus resultierende zeitdiskrete Struktur in der Teilchendynamik. Aus der selbstkonsistenten Lösung wird dann ein Stabilitätskriterium für jede Mode des Strahls abgeleitet. Die erhaltenen Ausdrücke erlauben auch eine Überlappung der Frequenzbänder im Spektrum des Strahls und liefern demzufolge über den gesamten Frequenzbereich gültige Aussagen.

Nachdem so die Grenzen der Stabilität festgelegt worden sind, erfolgt eine Beschreibung der Kühlung durch die Fokker-Planck-Gleichung. Die Berechnung ihrer Drift- und Diffusionskoeffizienten wird im Frequenzbereich durchgeführt. Auch sie betrachtet den Detektor und Kicker des Kühlsystems als lokale Objekte und beinhaltet somit die Taktung in der Kühlwechselwirkung. Die Fokker-Planck-Gleichung liefert eine statistische Beschreibung, die kollektive Effekte nicht einbezieht und daher implizit die Stabilität des Strahls voraussetzt. Die hieraus folgenden Vorhersagen über die Kühlung sind folglich nur innerhalb der hergeleiteten Stabilitätsgrenzen physikalisch sinnvoll. Daher wird am Ende geprüft, ob die ermittelten Parameter, mit denen das Kühlsystem am effizientesten arbeitet, verträglich sind mit der Stabilität des Strahls.

Abstract

This thesis provides a description of unbunched beam stochastic cooling in the framework of control theory. The main interest in the investigation is concentrated on the beam stability in an active cooling system. A stochastic cooling system must be considered as a closed-loop, similar to the feedback systems used to damp collective instabilities. These systems, which are able to act upon themselves, are potentially unstable and therefore their stability must be carefully analysed.

Assuming a linear transverse cooling interaction, the self-consistent solution for the beam motion is derived by means of a mode analysis of the collective beam motion. Furthermore the calculation treats the pick-up and kicker of the cooling system as localized objects which impose a discrete time structure on the dynamics of the beam particles. This solution then yields a criterion for the stability of each collective mode. The expressions which have been obtained also allow for overlapping frequency bands in the beam spectrum and thus are valid over the entire frequency range.

Having established the boundaries of stability in this way, the Fokker-Planck equation is used to describe the cooling process. The drift and diffusion coefficients are derived in the frequency domain taking into account the localization of pick-up and kicker and the sampled nature of the cooling interaction. The Fokker-Planck equation provides a purely statistical description, which does not include collective effects and thus a stable beam must be assumed. Hence the predictions about the cooling process following from the Fokker-Planck equation only make physical sense within the boundaries of beam stability. Finally it is verified that the parameters of the cooling system which give the best cooling results are compatible with the stability of the beam.

Contents

Introduction

Motivation

Storage rings aim to supply intense particle beams with long lifetimes and to bring them into collision with high luminosity. Hence small beam dimensions are desirable because they can improve both the lifetime and the luminosity. In this regard electron beams distinguish themselves from proton beams[1] because they have an inherent damping mechanism. The quantum-like energy loss, due to their synchrotron radiation, in combination with the acceleration in the rf-cavities hold the beam dimensions in an equilibrium state and thus make electron beams insensitive to small perturbations [1]. For proton beams this radiation damping is negligible for the beam energies which currently can be attained, so that already small excitations of the beam lead to a continuous increase of the beam dimensions. In the HERA proton ring, for example, typical emittance growth rates of $1\,\pi\,\mathrm{mm\,mrad/h}$ have been observed during beam collisions [2]. Because this emittance growth degrades the luminosity, various theories have been studied to explain this effect [3, 4]. At present, the dimensions of proton beams can only be reduced by means of an external system which provides an artificial damping mechanism. This active process of emittance reduction is called *beam cooling*.

The cooling of a particle beam can be understood as an increase of its phase-space density. This process concentrates the particles in the center of their distribution and thus reduces the phase-space volume which they occupy. This requires a local interaction in phase space which acts individually on the particles. In other words, the system has to resolve small fractions of the phase space of the beam in order to manipulate the internal phase-space structure.

In stochastic cooling systems this interaction happens by means of an external feedback loop, similar to the feedback systems used to damp collective instabilities [5, 6, 7, 8]. The basic idea is that a pickup detects from each particle the quantity to be reduced and a kicker feeds the amplified signals back to the particles with an appropriate phase shift to reduce the measured offsets. In order to act on the particles with their proper corrections the cooling system must be arranged in such a way that the time delay of the signals in the electronic components matches the transit time of the particles between pickup and kicker. However, due to their finite bandwidth, stochastic cooling systems are not able to resolve single particles so that the correction signals always contain the information of many particles. Besides the coherent self-interaction which provides damping a particle receives the signals from other particles being processed at the same time. In the case of random particle motions the latter produces an incoherent contribution in the cooling interaction which causes a diffusion. This diffusion counteracts the damping, and thus degrades the

[1]The same is true for the corresponding anti-particle beams.

cooling. In particular, the cooling system has to operate in such a way that a net cooling effect remains, which limits the system gain.

The description so far assumes random particle motion which strictly speaking only holds true until the first correction has been applied. The feedback of the cooling system introduces correlations among the particles which are manifested as a coherent beam motion causing a loss of the statistical independence of the particles. Different frequencies of the particles, however, lead to a decoherence of the collective motion and thus can destroy these correlations. For a sufficiently large frequency spread the phases will randomize within one revolution removing the correlations completely between successive cooling steps. This perfect phase mixing results in the shortest cooling time. Decays of correlations which take more than one revolution deteriorate the cooling because remaining correlations increase the diffusion effect. On the other hand, if the buildup of correlations occurs faster than their decay by the phase mixing, the correlations will continuously grow. Because then all beam particles participate in a collective motion, the resulting coherent interaction is many times stronger than the single particle self-interaction and therefore dominates the dynamics. In that case the collective particle motion arising from the correlations leads to instability of the beam.

In this picture stochastic cooling is divided into two competing processes, the cooling through the self-interaction and the diffusion. For a rigorous mathematical description which includes both effects one studies the particle density distribution in phase space. The time evolution of the phase space density reflects the dynamics of the cooling process and is usually derived from a Fokker-Planck equation. This approach, which is also used in this work, provides analytical expressions for the parameters which characterize the performance of the cooling system and allows quantitative predictions of the maximum attainable cooling rates.

The Fokker-Planck equation describes the cooling process at the microscopic particle level. Its derivation presumes the statistical independence of the individual particles and thus completely neglects the collective effects of the beam motion. The following remarks emphasize this point:

- Initially the Fokker-Planck equation determines the time evolution of the probability density of a single particle in phase space. Applying this result to all N particles of the beam and thus identifying the probability density with the phase space density of the beam requires the statistical independence of the particles.
- The calculations of its drift and diffusion coefficients use a perturbation expansion in a small parameter ϵ which measures the strength of the feedback force. This perturbation series converges only if the feedback force remains bounded which implies a stable beam motion.

Hence one obtains physically reasonable results from the Fokker-Planck description only within the stability boundaries of the beam. Since stochastic cooling systems close a feedback loop in which the beam acts upon itself, they are potentially unstable like any feedback system. Therefore a thorough stability analysis of the collective beam motion in cooling systems is a prerequisite for the applicability of the Fokker-Planck equation.

The existing reports about stochastic cooling either omit the verification of this requirement and implicitly assume a stable beam or consider simplified cases and thus obtain results which are valid only within certain limits [9]. For that reason the main interest in the investigation of this thesis is concentrated on the beam stability in an active cooling system. In

the following chapters transverse stochastic cooling of unbunched beams is studied for the case of a linear cooling interaction.

The analysis benefits from the distinct time scales underlying instabilities and stochastic cooling. Instabilities typically develop within milliseconds whereas the cooling times span a range from a few seconds to many hours. On the time scale of instabilities the internal phase space configuration of the beam only changes immaterially by means of the cooling interaction and therefore can be regarded as constant. Hence a separate treatment of the two processes becomes possible so that the beam stability is studied decoupled from the phase space cooling. The short time scale of instabilities also suggests an investigation in frequency domain. The collective motion of the beam is decomposed into the coherent beam modes and stability criteria are derived for each mode which allow predictions about the beam stability over the entire frequency range.

The method which is used to obtain these results differs from the usual treatment of instabilities by the Vlasov-theory. It is based on the theory of multi-bunch feedback systems which was derived from the control theory of discrete time signals [10]. This theory already includes the discrete time structure of the interaction originating in the localized pickup and kicker which is essential for a careful stability analysis.

For bunched beams, however, this method does not succeed in the same way due to the basically different longitudinal motions of particles in bunched and unbunched beams. Although first results for bunched beams have been obtained, they still necessitate further investigations and hence will not be presented in this work.

Outline

First, Chapter 1 reviews the theoretical and practical aspects of stochastic cooling. Chapter 2 introduces the basic terms and concepts of the control theory which become important in the following investigations. This formalism is used in Chapter 3 to derive the stability criteria of the coherent beam modes. These results determine the range within which the parameters of the cooling system preserve beam stability. Chapter 4 gives a mathematical description of the cooling process, including the calculations of the cooling parameters. Finally, the results of Chapter 4 are compared with the limits derived in Chapter 3 in order to verify their compatibility with the stability of the beam.

Chapter 1

Theory and Applications of Stochastic Cooling

1.1 Applications of Stochastic Cooling

Stochastic cooling was invented by S. van der Meer in 1968[1] but was only experimentally demonstrated seven years later. Since that time large improvements have been achieved in the technical realization of cooling systems, which has opened more and more new application fields for stochastic cooling. This section gives an overview of these practical uses.

Production of Intense Antiproton Beams

An important application of stochastic cooling is the accumulation of antiprotons which render an efficient operation of $p\bar{p}$ storage rings possible. The existing $p\bar{p}$ accelerators – the Super-Proton-Synchrotron (SPS) at CERN and the TEVATRON at Fermilab – have made considerable contributions to high-energy physics, in particular the discoveries of the Z^0 and $W^{\pm}$ bosons as well as of the top quark. A prerequisite for these successes had been intense antiproton beams without which the necessary luminosity could never had been delivered.

The production of antiprotons uses a high-energy proton beam which is directed at a metal target. The production rate of the antiprotons is however small and the delivered beam has a broad momentum spread and large transverse emittances. To obtain an intense antiproton beam the antiprotons are collected over a long time. This process is called accumulation and takes place in storage rings specially designed for that purpose.

Accumulation becomes possible by virtue of longitudinal stochastic cooling. The principle is based on the fact that the mean energy of the antiprotons differs from their storage energy in the accumulator ring. The energy difference is chosen such that a newly-injected antiproton beam does not affect the stored beam. Longitudinal stochastic cooling then adjusts the energies of the incoming antiprotons to the storage energy, and thus provides the longitudinal phase-space required for the following antiprotons. Since the antiprotons stay in these accumulator rings for a long time (up to 24 hours), they are, in addition, cooled in the longitudinal and transverse directions in order to preserve the increased phase-space densities over the period of accumulation.

[1] He published his idea for the first time in 1972 [11].

Examples of accumulator rings are the Antiproton-Accumulator (AA) and the Antiproton-Collector (ACOL) at CERN and the Debuncher/Accumulator-complex at Fermilab. These rings enhance the longitudinal phase-space density by a factor of $\gtrsim 10^4$, and the transverse phase-space densities by factors 10 to 100 [7].

Improvement of the Beam Properties

Stochastic cooling makes it feasible to generate narrowly-collimated and almost mono-energetic beams without any loss of particles. Longitudinal stochastic cooling systems allow a reduction of the energy spread of the beam and thus improve the energy resolution at the experiments. Transverse cooling decreases the horizontal and vertical beam emittances and therefore raises the luminosity. Hence stochastic cooling can substantially contribute to better experimental conditions.

In the Low-Energy-Antiproton-Ring (LEAR) at CERN, for example, transverse emittances of $\epsilon_{x,z} \lesssim 3\pi\,\mathrm{mm\,mrad}$ and a momentum spread of $\Delta p/p < 0.2\%$ were attained in a beam with $\sim 5 \cdot 10^{10}$ particles [12]. Owing to its stochastic cooling systems, LEAR can deliver high-quality antiproton beams for precision measurements.

Preservation of the Beam Quality

In stored beams various effects, e.g. intra-beam scattering, residual gas scattering or beam-beam interaction, can cause a growth of the transverse emittances and of the energy spread which in general results in particle loss. In this case stochastic cooling can be used to compensate the undesired increase, and thus preserves the beam quality during long storage times.

Especially ion storage rings profit from this process, e.g. the Experimental-Storage-Ring (ESR) at the GSI in Darmstadt, the Cooler-Synchrotron (COSY) at the KfK Jülich, the Test-Storage-Ring (TSR) at the MPI in Heidelberg, CELSIUS in Uppsala and ASTRID in Aarhus, to mention just a few.

Bunched Beam Cooling

The application fields considered so far all refer to the cooling of unbunched beam. Indeed concrete efforts exist to apply stochastic cooling also to bunched beams. Fermilab aims at cooling the bunched proton and antiproton beam in the TEVATRON, both horizontally and vertically. The planned systems are to counteract the emittance growth of the beams which is mainly caused by power-supply ripples and electronic noise [13]. The idea is to raise the luminosity lifetime so that the beams can be stored over longer periods. Since in that case more time becomes available for the antiproton accumulation, one ends up with more intense antiproton beams. On the other hand, the beams have to be replaced less frequently so that altogether the useful time for beam collisions increases. In that way one hopes to double the integrated luminosity [13].

First tests have already been carried out with a vertical cooling system for the proton beam, but measurable changes in the emittance growth rate could not be observed so far [13]. Before stochastic cooling can efficiently be applied to bunched beams, a lot more research and development will be necessary.

1.2 Theoretical Description of Stochastic Cooling

The theoretical formulation of stochastic cooling can follow various ways. One possibility is to look at the process purely in the time domain. This provides a very intuitive picture of the cooling but does not allow precise quantitative predictions. On the other hand, one can analyse the beam dynamics in the frequency domain by means of the spectrum generated by the particles. This more rigorous mathematical treatment yields reliable predictions about the cooling process. Both formulations represent statistical descriptions which rely on the signal fluctuations due to the discreteness of the beam particles. In this microscopic view, the collective beam motion is completely disregarded.

Another approach to stochastic cooling is given by the kinetic theory. It investigates the time evolution of the $1-, 2-, \ldots, N-$particle distribution functions and hence takes into account the correlations among the particles. Thus predictions about collective effects become possible. On the other hand, this method requires a substantial mathematical effort and one obtains analytical solutions only for simple systems. Here, we will not further pursue this path. More detailed information can be found in [7, 8].

In the next two sections we elaborate on the sample picture in the time domain and the description by a Fokker-Planck equation which in its relevant parts is performed in the frequency domain.

The Sample Picture

We now discuss the formulation of stochastic cooling in the *sample picture*, as it has been developed in [6]. For this purpose we consider a transverse cooling system for unbunched beams. However the major concern does not aim at detailed mathematical derivations but much more at a discussion of the assumptions underlying this description. For the most part the argumentation follows [6, 7] where further details can also be found.

The formulation is based on the idea of dividing the beam into samples. Because of its finite bandwidth, W, a cooling system cannot resolve the individual particles in a dense beam and thus always processes many particles simultaneously which in each case define a sample. The size of the sample, i.e. its number of particles, is given by $N_S = N/2WT_0$. Here, N is the total number of particles in the beam and T_0 denotes the nominal revolution time. The larger the bandwidth W, the smaller the samples processed by the system, and the more distinctly the individual signal contribution of each sample particle will emerge. In order to come closer to the ideal case in which each particle is cooled separately, we have to make the system bandwidth as large as possible.

Since all sample particles contribute to the correction of a particular sample particle i, we can write its displacement $\check{x}_i$ after the correction as

$$\check{x}_i = x_i - \lambda \sum_{j=1}^{N_S} x_j, \tag{1.1}$$

where λ is the strength with which the measured particle displacements are fed back. Hence it follows, for the difference of x_i^2 before and after the correction

$$\Delta x_i^2 = -2\lambda \sum_{j=1}^{N_S} x_i x_j + \lambda^2 \sum_{j=1}^{N_S} \sum_{j'=1}^{N_S} x_j x_{j'}. \tag{1.2}$$

Before the first correction the sample particles can be considered as statistically independent so that averaging over their displacements yields $\langle x_j x_{j'}\rangle = \delta_{jj'}$. So we obtain for Eq. (1.2)

$$\langle \Delta x^2\rangle = -2\lambda\langle x^2\rangle + \lambda^2 N_S \langle x^2\rangle. \tag{1.3}$$

The first term arises from the self-interaction of the particles. It is the *coherent* contribution which effects the intrinsic cooling. The second term describes the impact of the other sample particles and represents the *incoherent* contribution. It results in a diffusion which increases the amplitudes of the particle motions, and thus can be interpreted as a heating of the beam.

The change (1.3) is valid only for the first correction since after this correction the particles are correlated and the assumption $\langle x_j x_{j'}\rangle = \delta_{jj'}$ is no longer justified. According to Eq. (1.1), we can write after the correction

$$\check{x}_i \check{x}_j = \left(x_i - \lambda \sum_{k=1}^{N_S} x_k\right)\left(x_j - \lambda \sum_{k'=1}^{N_S} x_{k'}\right).$$

Even if $\langle x_i x_j\rangle = 0$ for $i \neq j$ is satisfied before the correction, the corresponding expression $\langle \check{x}_i \check{x}_j\rangle$ after the correction contains non-zero terms of the form $-\lambda\langle x_i^2\rangle, -\lambda\langle x_j^2\rangle, \ldots$ expressing the correlations among the particles.

On the other hand, different revolution frequencies of the particles can destroy these correlations. In the sample picture this process is called *mixing* and illustrated by a change of the sample population rendering the sample particles again statistically independent. Hence mixing requires that the decay of correlations occurs faster than their build-up by the cooling interaction, otherwise the correlations would continuously increase and the collective motion of the particles would dominate. In that case the coherent particle motions can cause an instability of the beam.

The sample picture describes the correlations by a constant, time-independent mixing factor which serves as a measure of how fast the samples are rearranged. The time evolution of the displacements x_i, and thus the dynamics of the particles, is disregarded in this picture. The initial decrement (1.3) is transferred to all successive cooling steps, thereby neglecting the correlations which are introduced by the cooling process. This can be seen clearly in the calculation of the cooling rate where the time evolution of the mean-squared amplitude is deduced from the initial change (1.3) obtained from statistically independent particles,

$$\frac{d}{dt}\langle x^2\rangle \longrightarrow \frac{\langle \Delta x^2\rangle}{T_0}.$$

The stochastic cooling formulation in the sample picture entirely relies on the mixing assumption, i.e. the fact that the reorganization of the samples is guaranteed. Based on this assumption, the description disregards both the individual particle motions and the collective motion of the beam. Consequently, beam stability is an indispensable prerequisite for the validity of the mixing assumption.
Finally, we quote the expression which the sample picture yields for the cooling rate [6]

$$\frac{1}{\tau} = \frac{W}{N}\left[2g(1-\tilde{M}^{-2}) - g^2(M+U)\right]. \tag{1.4}$$

Since the derivation does not take into account the particle motions, the dynamic effects of the cooling process must explicitly be introduced by means of empirical arguments.

In detail, we find in Eq. (1.4):

- The factor $(1-\tilde{M}^{-2})$ describes the undesirable migration of particles into other samples on their way from the pickup to the kicker. This *unwanted mixing* only effects the self-interaction of the particles and degrades the coherent contribution.

- The desired mixing of the sample between the kicker and pickup is represented by the mixing factor M and interpreted as a weighting of the incoherent contribution. The worse the sample reorganization between successive cooling steps, the larger the mixing factor M, and the more pronounced the incoherent contribution to the cooling rate will be.

- The quantity U models the electronic noise in the cooling system and leads to an additional enhancement of the incoherent contribution.

- Once more the importance of the bandwidth W is stressed. We recognize that an increase of bandwidth results in a faster cooling rate and thus improves the cooling. For that reason the bandwidth is of fundamental concern in any cooling system.

Newer cooling systems have bandwidths of up to 4 GHz and typically cover one of the frequency bands from 1-2 GHz, 2-4 GHz or 4-8 GHz. The number of particles in the beams varies in these systems between 10^8 and 10^{12}. LEAR, for example, stores antiproton beams of $\sim 10^{10}$ particles. The cooling times attained in these systems reach from a few seconds up to some hours.

The Fokker-Planck Equation

A more rigorous mathematical representation of stochastic cooling which provides exact quantitative predictions about the cooling process is given by the Fokker-Planck equation. In this section the basic ideas of this formulation are compiled, intending to disclose the approximations which enter into the description. More details and the results following from the Fokker-Planck treatment of stochastic cooling can be found in Chapter 4. Information going beyond that can be found in [7, 8, 14, 15].

The Fokker-Planck equation determines, for each particle, the time development of its probability density in the quantity being cooled. In principle these densities allow the calculation of all statistical moments of the relevant quantity and therefore provide a complete description of the cooling process. For statistically independent particles the probability densities are all the same and can be identified with the corresponding distribution function of the beam [8, 14].

To illustrate this we once more consider transverse cooling of the betatron motions. In this case an appropriate variable to describe the cooling process is the action I of each particle. In connection with this variable we define a density $\rho(I,t)$ such that $\rho(I,t)dI$ gives the number of particles with actions between I and $I+dI$ at a time t. The density $\rho(I,t)$ is connected with the probability density $\psi(I,t)$ of the independent particles by the relation $\rho(I,t) = N\psi(I,t)$ and its time development is determined by the following Fokker-Planck equation [15]

$$\frac{\partial}{\partial t}\rho(I,t) = -\frac{\partial}{\partial I}\left\{F(I)\rho(I,t) - \frac{1}{2}D(I)\frac{\partial}{\partial I}\rho(I,t)\right\}.$$

The drift and diffusion coefficients $F(I)$ and $D(I)$ are obtained from the expressions

$$F(I) = \Big\langle \frac{\Delta I}{\Delta t} \Big\rangle \qquad \text{and} \qquad D(I) = \Big\langle \frac{(\Delta I)^2}{\Delta t} \Big\rangle,$$

where ΔI denotes the change of action in the time interval Δt, and the square brackets indicate an average over the initial conditions. The calculation of ΔI appears difficult because the instantaneous change $\dot{I}$ of the action depends on the actions of all N particles in the beam. Representing the cooling interaction by an appropriate function $G(I_1, \ldots, I_N, t)$, we can write

$$\dot{I} = G(I_1, \ldots, I_N, t).$$

Integration of this expression over the time interval Δt does not yield the desired value ΔI because the integrand itself depends on the yet unknown actions,

$$\Delta I = \int_0^{\Delta t} dt \; G(I_1, \ldots, I_N, t).$$

To overcome this difficulty we are forced to use a perturbation expansion and hence substitute in the integrand the unperturbed actions I^0 which are derived from the known zero-order particle motions,

$$\Delta I = \int_0^{\Delta t} dt \; G(I_1^0, \ldots, I_N^0, t).$$

In general this step allows the evaluation of the integral and with it the determination of the drift and diffusion coefficients necessary to solve the Fokker-Planck equation. This proceeding corresponds to Picard-Lindelöf's iteration method which converges only if the function $G(I_1, \ldots, I_N, t)$ satisfies a Lipschitz condition [16]. In the present case this implies that the feedback force of the cooling interaction must remain bounded which can only be ensured by a stable beam motion.

It should be mentioned that the same approximation is made in the sample picture by applying the first correction (1.3) valid only for uncorrelated particles to all successive cooling steps. By that the modification of the particle motions is neglected, thus assuming that the particles still move along their initial, unperturbed trajectories. Therefore the above considerations hold true for the sample picture as well.

Since Chapter 4 will give full details of the cooling description by a Fokker-Planck equation, we restrict ourselves here to some general remarks. The information about cooling interaction and the structure of the cooling system are contained in the drift and diffusion coefficients so that their derivations include

- the individual particle motions,
- the positions of pickup and kicker and
- the signal transfer through the electronic components.

The calculations benefit from a treatment in the frequency domain, as in [17]. There it has been shown that the particles can be discriminated by their frequencies, and over long times only interact via common frequencies in their spectra (see also Sect. 2.6). The representation in the frequency domain allows a clearer interpretation of the inter-particle correlations

than a time domain description, and furthermore quantitative predictions about the mixing become possible.

A particle with tune Q and revolution frequency ω generates, at a pickup, a spectrum of lines at the frequencies $(m+Q)\omega$ with $m = 0, \pm 1, \pm 2, \ldots$ and can be coherently excited only at these frequencies (see Sect. 2.5). The number of particles producing, in their spectra, the same frequencies $(m + Q)\omega$, determines the strength of the diffusion and is a measure how well mixing occurs. More generally, all dynamic properties of the cooling process which had to be introduced empirically into the sample picture emerge automatically from the Fokker-Planck description. This follows from the consideration of the particle motions and the localization of the pickup and kicker in the calculations of the drift and diffusion coefficients, and will become apparent in the discussion of the results in the Sections 4.5.3 and 4.5.4.

The cooling descriptions presented above have in common that they are restricted to the microscopic interactions between individual particles without involving the coherent beam motion. They only consider the long-term behaviour of the beam which is governed by the cooling, and presume beam stability in the active cooling system. The major deficiency of these descriptions is the absence of the necessary stability analysis so that they rely on unfounded assumptions. This thesis could remove these shortcomings. Here, the existing descriptions have been extended by the stability investigation omitted so far, providing them with a solid physical and mathematical foundation.

Especially when stochastic cooling is applied to complex accelerators, such as HERA, the TEVATRON or the LHC, the collective beam dynamics must be thoroughly understood. An over-simplified description cannot rule out that problems will later arise from coherent beam signals, preventing the operation of the system. This method has already proven true in the realization of the feedback systems at DESY whose reliability is largely due to the fact that their conceptional designs are based on detailed theoretical investigations.

Chapter 2

Stability in Feedback Loops

2.1 Stochastic Cooling Systems as Closed Loops

The mathematical description of stochastic cooling involves two different aspects. One is mainly interested in predictions about the cooling performance which are usually derived within a model which assumes statistically independent particles and thus neglects any correlations among them. Since stochastic cooling systems close a loop in which the beam can act upon itself (see Fig. 2-1), they are potentially unstable and therefor their stability must be analysed. In this chapter we develop the methods which will later be used to investigate the collective behaviour of the particles.

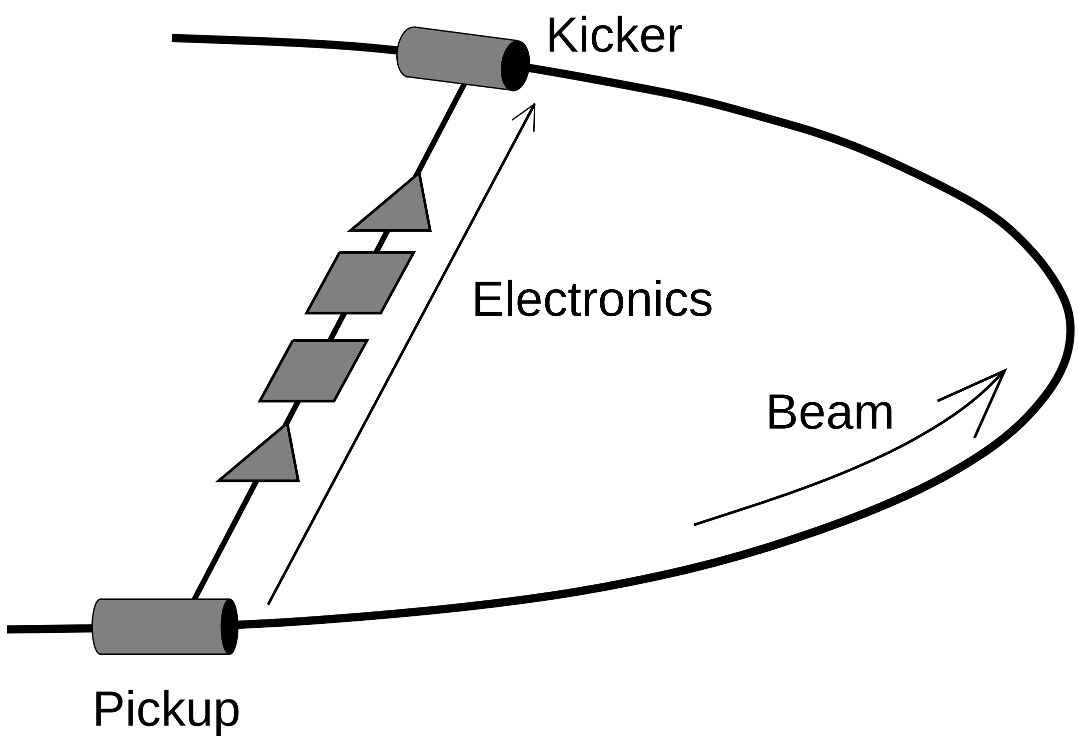

Figure 2-1: *Schematic view of a cooling system.*

In principle stochastic cooling systems operate in the following way:

- The particles generate signals at the pickup which contain the information necessary for their corrections.

- The signals are processed in the electronics of the system which in general changes their amplitude and phase.
- The kicker feeds the modified signals back to the beam in order to reduce the measured offsets.

Later in this chapter we will see that a thorough investigation of the stability has to consider the pickup and kicker as *localized* objects, leading to a discrete time structure of the signals.

In this thesis we study the stability within the framework of multi-bunch feedback theory which has been developed from the control theory of discrete time signals [10, 18]. In the following sections we introduce the basic concepts of the theory by means of simple examples.

2.2 The Transfer Function

The transfer function describes the relation between cause and consequence of an interaction. In Fig. 2-2, for example, an external kicker excites betatron oscillations in a beam. The cause in this case is the kick $g(t)$ applied to the beam (at the kicker) and the consequence is the displacement $y(t)$ of the beam due to the resulting oscillation.

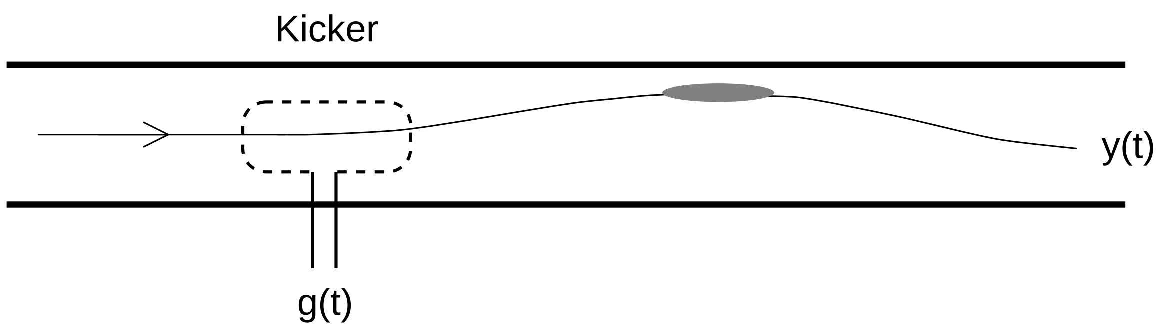

Figure 2-2: *Beam excitation by an external force.*

For a linear relation between cause $g(t)$ and consequence $y(t)$ we write

$$y(t) = \int_{-\infty}^{+\infty} dt' \; G(t,t') g(t') \tag{2.1}$$

which defines the impulse response $G(t,t')$. Having changed variables according to $\{t,t'\} \longrightarrow \{t, t-t'\}$, we stipulate the additional properties of the impulse response G:

$$\begin{array}{lcl} (1)\ G \text{ is stationary} & \Longleftrightarrow & G(t,t-t') = G(t-t') \\ (2)\ G \text{ is causal} & \Longleftrightarrow & G(t-t') \equiv 0 \quad \text{for} \quad t-t' \leq 0 \\ (3)\ G \text{ is real} & & \end{array}$$

Using Eq. (A.5), the general Fourier transformation (2.3) of Eq. (2.1) yields the product

$$\widetilde{y}(w) = \widetilde{G}(w)\widetilde{g}(w).$$

The Fourier transform $\widetilde{G}(w)$ of the impulse response is called *transfer function*. Since the time functions $y(t)$, $G(t)$ and $g(t)$ are real, it follows that

$$\widetilde{y}^*(w) = \widetilde{y}(-w^*) \ , \quad \widetilde{G}^*(w) = \widetilde{G}(-w^*) \quad \text{and} \quad \widetilde{g}^*(w) = \widetilde{g}(-w^*).$$

2.3 The General Fourier Transformation

In accelerator physics it is reasonable to assume that the relevant functions do not grow faster in time than

$$|f(t)| \leq M e^{\alpha t} \quad \text{for} \quad t \geq 0 \tag{2.2}$$

with real constants $M, \alpha > 0$. Allowing for complex frequencies w, the general Fourier transform of $f(t)$ is defined by [19]

$$\tilde{f}(w) = \frac{1}{2\pi} \int_0^\infty dt\, f(t) e^{-iwt}. \tag{2.3}$$

$\tilde{f}(w)$ is an analytical function at least in the lower w-plane for $\mathsf{Im}\, w < -\alpha$. The inverse transformation is given by

$$f(t) = \int_C dw\, \tilde{f}(w) e^{iwt}. \tag{2.4}$$

The path C has to lie in the analytical region of $\tilde{f}(w)$. It can be chosen as a straight line parallel to the real axis with $\mathsf{Im}\, w < -\alpha$ and thus is below the singularities of $\tilde{f}(w)$. For our purposes we need to consider only *simple poles* (see Fig. 2-3).

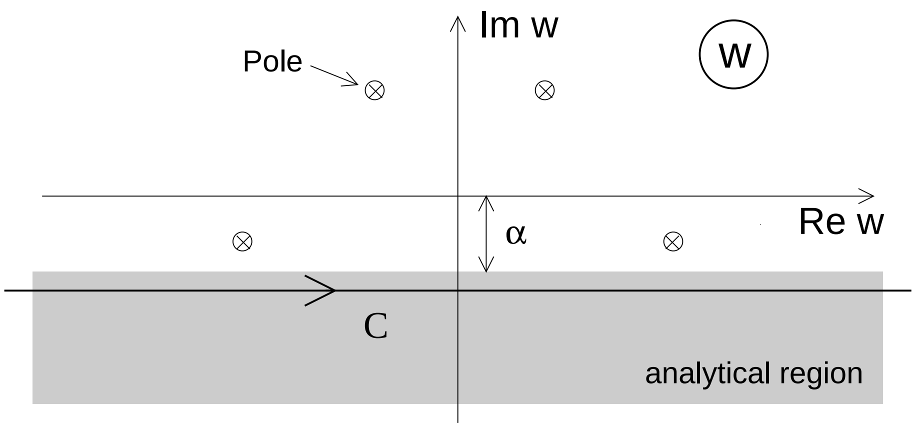

Figure 2-3: *The integral contour C of the inverse transformation.*

The time evolution of $f(t)$ is completely determined by the poles of $\tilde{f}(w)$. Evaluating the integral (2.4) with the residue theorem yields [19]

$$f(t) = 2\pi i \sum_k \mathrm{res}_{w_k} \left\{ \tilde{f}(w) \right\} e^{iw_k t} \tag{2.5}$$

where the sum extends over all poles w_k of $\tilde{f}(w)$. The poles result in oscillating terms with exponentially growing or decreasing amplitudes depending on the sign of the imaginary part $\mathsf{Im}\, w_k$. In the upper w-plane the imaginary parts are positive and give damped solutions, since with $w_k = \omega_k + i\alpha_k$

$$f(t) \sim e^{iw_k t} \sim e^{i\omega_k t} e^{-\alpha_k t} \longrightarrow 0 \qquad \text{for} \qquad t \longrightarrow \infty \quad \text{and} \quad \alpha_k > 0.$$

Correspondingly poles in the lower w-plane which have $\mathsf{Im}\, w_k < 0$ describe growing, unstable solutions.

Another advantage of the general Fourier transformation appears in the solution of linear differential equations: the time derivatives of a function transform into simple algebraic expressions in the frequency domain which moreover include the initial conditions, e.g.

$$\begin{aligned}
\left[\widetilde{\dot{f}(t)}\right] &= iw\tilde{f}(w) - \frac{1}{2\pi}f(0) \\
\left[\widetilde{\ddot{f}(t)}\right] &= -w^2\tilde{f}(w) - \frac{iw}{2\pi}f(0) - \frac{1}{2\pi}\dot{f}(0)
\end{aligned}$$

where $f(0)$ and $\dot{f}(0)$ denote the values of $f(t)$ and $\dot{f}(t)$ at the time $t = 0$. The general expression together with some other useful properties of the general Fourier transformation can be found in Appendix A, and in example B.1 the formalism is applied to a free harmonic oscillator.

2.4 The Feedback Mechanism

The concept of transfer functions introduced in Section 2.2 also applies if the cause which modifies the beam motion originates in the beam itself, i.e. if the beam indirectly acts upon itself. In the mechanism of such a feedback the concept of impedance plays an important part. The impedance describes the relation between an excitation and the resulting response in frequency domain, analogous to the transfer function in time domain. Impedance and transfer function are connected through the general Fourier transformation. The general feedback mechanism splits into two basic steps:

(i) The collective motion of the beam generates electromagnetic fields through the impedances of the storage ring.

(ii) These electromagnetic fields act upon the beam and thus modify its motion.

Given the right phase relation between beam motion and reacting fields the oscillation amplitude of the beam will continuously grow, and the beam motion becomes unstable. In order to predict the stability behaviour of a beam in a feedback loop, we need a self-consistent description of the beam motion. The following simple example will clarify this further.

For that purpose we consider a harmonic oscillator with frequency Ω_0 which acts upon itself. Being stationary for times $t \leq 0$, the oscillator is excited by a δ-pulse $g(t) = A\delta(t)$. The equation of motion can be written as

$$\ddot{x}(t) + \Omega_0^2\, x(t) = F[x](t) + g(t).$$

Assuming a linear response, the reacting force $F[x](t)$ reads (see Eq. (2.1))

$$F[x](t) = \int_0^\infty dt'\; G(t-t')x(t')$$

where $G(t-t')$ denotes the transfer function. Hence follows

$$\ddot{x}(t) + \Omega_0^2\, x(t) = \int_0^\infty dt'\; G(t-t')x(t') + g(t). \tag{2.6}$$

The general Fourier transformation of Eq. (2.6) yields

$$(-w^2 + \Omega_0^2)\,\widetilde{x}(w) = \widetilde{G}(w)\widetilde{x}(w) + \widetilde{g}(w).$$

The problem of the self-consistent description of the motion reduces to the solution of an algebraic equation in the frequency domain. For the present case we easily obtain

$$\widetilde{x}(w) = \frac{\widetilde{g}(w)}{-w^2 + \Omega_0^2 - \widetilde{G}(w)}. \tag{2.7}$$

Since the time behaviour of $x(t)$ is completely determined by the poles of $\widetilde{x}(w)$ (see Sect. 2.3), we must find the zeros in the denominator of Eq. (2.7)

$$-w^2 + \Omega_0^2 = \widetilde{G}(w).$$

This is demonstrated in the example B.2 for a given impedance $\widetilde{G}(w)$.

2.5 Discrete Time Signals

The formalism developed so far describes continuous self-interactions, such as through the broad-band impedance of the storage ring. In feedback systems, however, the interaction takes place via pickup and kicker which are localized objects in the storage ring. Hence the particles generate signal pulses at the pickup and they sample the forces at the kicker with the revolution time which results in a *discrete* time structure of the interaction.

We will illustrate this for the transverse signal at a pickup produced by a particle which executes betatron oscillations. Because the particle passes the pickup only once per turn, it does not produce a continuous signal but a series of amplitude-modulated δ-pulses separated by its revolution time T (see Fig. 2-4).

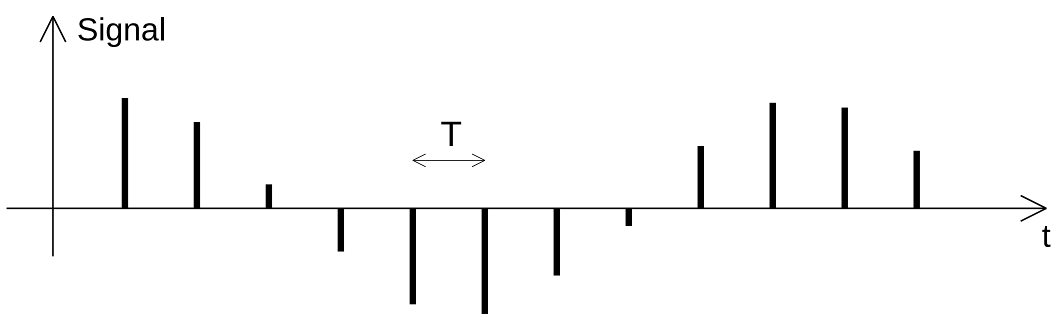

Figure 2-4: *The discrete time signal generated by a transversally oscillating particle in a localized pickup.*

Assuming that for $t = 0$ the particle is at the pickup, we write its signal $S(t)$ as

$$S(t) = x(t) \sum_{(k)} T\delta(t - kT) = \sum_{(k)} x(kT)\, T\delta(t - kT) \quad \text{where} \quad \sum_{(k)} \equiv \sum_{k=-\infty}^{+\infty}. \tag{2.8}$$

$x(t)$ describes the betatron oscillation of the particle and T denotes its revolution time. The general Fourier transformation of $S(t)$ yields

$$\widetilde{S}(w) = \frac{1}{2\pi}\int_0^\infty dt\, S(t)e^{-iwt} = \frac{1}{2\pi}\int_0^\infty dt\, \sum_{(k)} x(kT)T\delta(t - kT)e^{-iwt} = \frac{T}{2\pi}\sum_{k=0}^{\infty} x(kT)e^{-iwkT}. \tag{2.9}$$

We express the displacement $x(kT)$ by its Fourier transformation

$$x(kT) = \int_C dw' \, \widetilde{x}(w') e^{iw'kT} \tag{2.10}$$

and thus obtain

$$\widetilde{S}(w) = \frac{1}{\omega} \sum_{k=0}^{\infty} \int_C dw' \, \widetilde{x}(w') e^{iw'kT} e^{-iwkT} = \int_C dw' \, \widetilde{x}(w') \frac{1}{\omega} \sum_{(k)} e^{i(w'-w)kT} \tag{2.11}$$

where $\omega = 2\pi/T$ is the revolution frequency of the particle. Since $\widetilde{x}(w')$ is analytic in the lower w-plane, the integral in Eq. (2.10) does not contribute if $k < 0$, and thus the summation in Eq. (2.11) can be extended over all values k. Using Poission's formula [19]

$$\sum_{(k)} e^{i(w'-w)kT} = \omega \sum_{(m)} \delta(w' - w - m\omega)$$

we find

$$\widetilde{S}(w) = \int_C dw' \, \widetilde{x}(w') \sum_{(m)} \delta(w' - w - m\omega) = \sum_{(m)} \widetilde{x}(w + m\omega).$$

Here we define the periodic function $\hat{x}(w)$ by

$$\hat{x}(w) = \sum_{(m)} \widetilde{x}(w + m\omega).$$

which has the period ω, since for any integer l

$$\hat{x}(w + l\omega) = \sum_{m=-\infty}^{\infty} \widetilde{x}(w + [m + l]\omega) = \sum_{k=-\infty}^{\infty} \widetilde{x}(w + k\omega) = \hat{x}(w)$$

where $k = m + l$ has been substituted.

The spectrum of a particle which generates a pulsed signal at a pickup in time with its revolution time T consists of an infinite series of lines separated by its revolution frequency ω. An equivalent result follows for a particle with revolution frequency ω which samples the force at a localized kicker. Given that the particle oscillates with the betatron frequency $\Omega = \mathsf{Re}\, w$, it can be excited coherently at any frequency $\Omega + m\omega$ with $m = 0, \pm 1, \pm 2, \ldots$.

The single particle results can easily be extended to N particles equally distributed around the storage ring. Since in this case the signals at the pickup are separated by the time difference $\Delta t = T/N$, the signal spectrum is periodic with the frequency $2\pi/\Delta t = 2\pi/(T/N) = N\omega$. For the same reasons a kicker can excite a coherent oscillation of the N particles only at frequencies $\Omega + lN\omega$ with $l = 0, \pm 1, \pm 2, \ldots$.

2.6 Overlapping Frequency Bands

In the previous section we have assumed equal revolution frequencies which is unlikely for the particles in an unbunched beam. Owing to the momentum spread of the particles which is always present in an unbunched beam, the revolution frequencies are spread over an interval. The signal spectrum generated by the beam particles at a pickup now consists of a series of *frequency bands* which mirror the frequency distribution of the particles. The width of the bands increases to higher frequency so that beyond a certain frequency they overlap.

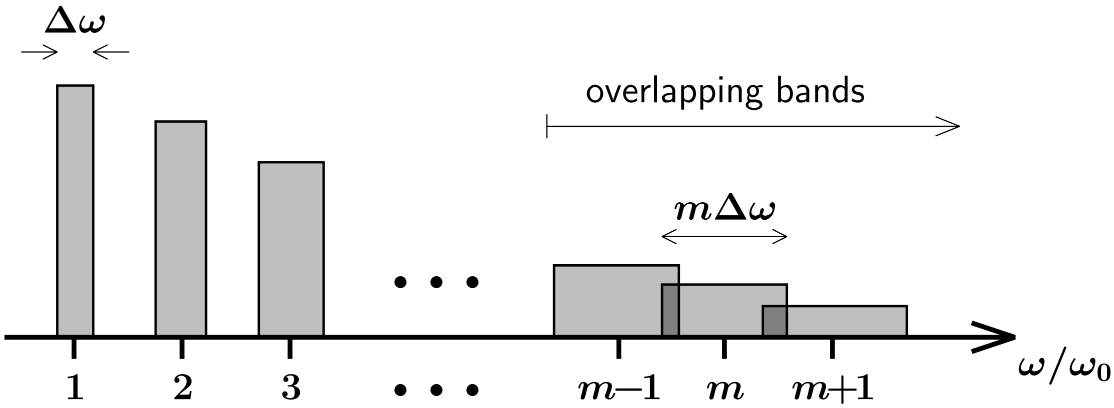

Figure 2-5: *Band spectrum for a rectangular frequency distribution.*

As an example we will discuss the spectrum of the longitudinal beam signal. For the sake of simplicity we assume a rectangular distribution of the revolution frequencies with a center frequency ω_0. So the frequencies are equally spread over the interval $\Delta\omega = \omega_{max} - \omega_{min}$ where ω_{max} and ω_{min} denote the upper and lower cut-off frequency of the distribution.

A particle j of the distribution with a revolution frequency ω_j generates a spectrum of lines at the revolution harmonics $m\omega_j$, $m = 0, \pm1, \pm2, \ldots$ (see Sect. 2.5). Consequently the lines for the lower and upper cut-off frequency of the distribution appear at the frequencies $m\omega_{min}$ and $m\omega_{max}$ respectively, so that the width of the revolution band m is given by $m\omega_{max} - m\omega_{min} = m(\omega_{max} - \omega_{min}) = m\Delta\omega$. Thus the bands become broader for increasing $|m|$. Since each band contains the same number of particles, the amplitude of the bands decreases as their width increases. This behaviour is shown in Fig. 2-5.

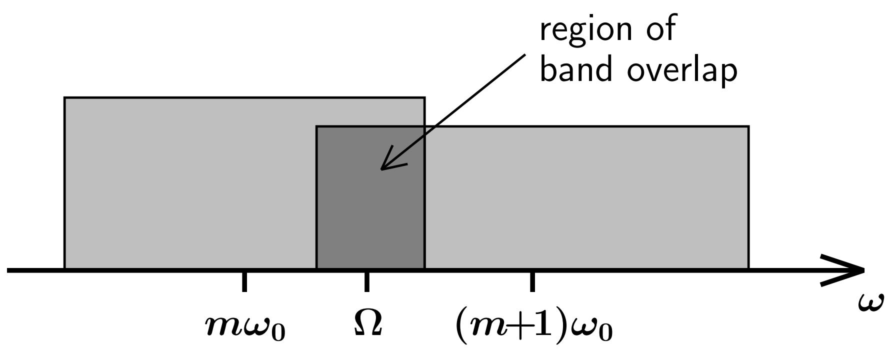

Figure 2-6: *Two overlapping bands with harmonic numbers m and $m+1$. In the overlap region revolution frequencies ω_j and $\omega_{j'}$ exist which satisfy $\Omega = m\omega_j = (m+1)\omega_{j'}$.*

For sufficiently high harmonic numbers m the width of the bands will satisfy $m\Delta\omega \gtrsim \omega_0$, and hence adjacent frequency bands will overlap (see Fig. 2-5). In this region of band overlap, particles with different revolution frequencies $\omega_j, \omega_{j'}, \omega_{j''}, \ldots$ from different revolution bands $m, m', m'', \ldots$ will contribute to the spectrum at the same frequency Ω if their harmonics fulfil the resonance condition $\Omega = m\omega_j = m'\omega_{j'} = m''\omega_{j''} = \ldots$. In this case the particles can no longer be distinguished by their revolution frequencies ω_j. Fig. 2-6 shows the overlap of two adjacent bands.

An analogous result follows for a localized force which the beam particles sample with their revolution times T_j. A periodically changing force with a frequency Ω falling into the overlap region of the bands will coherently excite particle with different revolution frequen-

cies $\omega_j, \omega_{j'}, \omega_{j''}, \ldots$ in different revolution bands $m, m', m'', \ldots$ if they meet the resonance condition $\Omega = m\omega_j = m'\omega_{j'} = m''\omega_{j''} = \ldots$.

Thus, in feedback systems the motion of particles with different revolution frequencies can couple through overlapping frequency bands. A particle with frequency ω_j generating signals at frequencies $\Omega = m\omega_j$ can act coherently on a particle with frequency $\omega_{j'}$, provided that the frequency bands m and m' (which satisfy the condition $m\omega_j = \Omega = m'\omega_{j'}$) overlap within the bandwidth of the system.

Chapter 3

Stability of the Unbunched Beam

3.1 The Existing Treatment of the Problem

The formalism developed in the previous chapter is now used to investigate the stability of an unbunched beam subjected to transverse stochastic cooling. To this end we must find a self-consistent description of the beam motion which treats the pickup and kicker as localized objects. In Section 2.6, we saw that this localization together with a frequency distribution in the beam leads to a band spectrum which extends over the entire frequency range. The width of the bands increases to higher frequencies, and beyond a certain frequency the bands overlap. The feedback interaction in the region of overlapping frequency bands couples the motions of almost all particles, complicating the derivation of the self-consistent solution of the beam motion. Because of this, most of the work on stochastic cooling disregards the localization of pickup and kicker, and thus obtains predictions about the beam stability which are only valid in the frequency range of non-overlapping bands [6, 9, 20]. Since the stochastic cooling operation profits to a certain degree from these overlapping bands, the results have only limited physical relevance. Although self-consistent solutions have been derived which thoroughly take into account the localization [21, 22], the beam stability in such a cooling system has not yet been investigated.

In this chapter, we present a careful analysis of the beam motion in order to predict the stability of the beam. First, we derive the equation of motion of a single particle undergoing stochastic cooling: the driving term in this equation contains the feedback force at the kicker which depends on the displacements of all particles at the pickup. For the stability analysis the decisive factor is not the individual particle motion, but the *collective* motion of all particles which is determined by averaged macroscopic quantities of the beam. Summing the single particle equations over all beam particles and ignoring the discreteness of the particles, we obtain equations describing the motion of a continuous beam. These equations are then solved by means of the methods discussed in the previous section, resulting in the self-consistent solution finally used to derive the stability criteria of the beam.

3.2 Particle Motion in an Unbunched Beam

We consider a single particle which is freely circulating in the longitudinal direction with a revolution frequency ω. If θ^0 denotes the initial azimuth at the time $t = 0$ (see Fig. 3-1), the azimuthal coordinate $\theta(t)$ of the particle can be written as $\theta(t) = \omega t + \theta^0$.

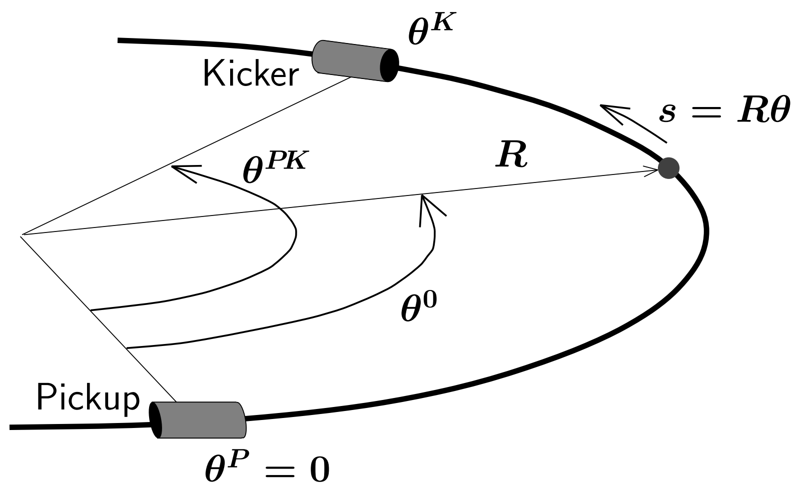

Figure 3-1: *The longitudinal coordinates of a particle.*

Assuming uncoupled horizontal and vertical motions of the particle, we need not distinguish between the planes and we will refer to both as simply transverse motion. To describe the transverse motion we define the quasi-time τ by [10]

$$\tau(t) = \frac{\varphi(s[t])}{\Omega} \tag{3.1}$$

where Ω denotes the betatron frequency. $\varphi(s[t])$ describes the betatron phase advance of the particle as a function of its orbit coordinate $s[t]$. The reference point is chosen such that $\tau(0) = 0$. Since the betatron phase advance φ of a particle is the same for each revolution, it follows

$$\tau(t + T) = \tau(t) + T \tag{3.2}$$

with the revolution time T. It can also be shown that

$$\frac{d\tau}{dt} = \frac{\omega}{\Omega} \frac{R}{\beta(s)} \tag{3.3}$$

where $\beta(s)$ is the beta function and R the radius of the accelerator.

It is convenient to express the transverse displacement $X(\tau)$ of the particle in Courant-Snyder variables which are defined by

$$x(\tau[t]) = \frac{X(\tau[t])}{\beta(s[t])}$$

since then the transverse motion of the particle obeys the equation of motion of a harmonic oscillator.

3.3 The Signal at a Pickup

We now derive the signal which the beam particles generate in a pickup at the position $\theta^P = 0$. The particle j passes the pickup at times $t_k = t_j^0 + kT_j$ with $k = 0, \pm1, \pm2, \ldots$ where

t_j^0 is the time it takes to cover its initial azimuth θ_j^0, i.e. $\theta_j^0 = \omega_j t_j^0$. Hence the signal $S(t)$ at the pickup can be written as

$$S(t) = \sqrt{\beta_P} \sum_{j=1}^{N} x_j\Big(\tau_j(t - t_j^0)\Big) \sum_{(k)} T_j \, \delta(t - t_j^0 - kT_j). \tag{3.4}$$

Here, β_P refers to the value of the beta function at the pickup, and the notation (k) implies a summation over all integers,

$$\sum_{(k)} \equiv \sum_{k=-\infty}^{+\infty} .$$

Writing the time dependence of x_j in this way, we explicitly take into account the betatron phase advance which corresponds to the initial azimuthal distance θ_j^0 of the particles from the pickup, and thus ensure that the transverse displacements of the particles at the pickup are summed with the proper phase. Using the periodicity (3.2), we obtain for Eq. (3.4)

$$\begin{aligned} S(t) &= \sqrt{\beta_P} \sum_{j=1}^{N} \sum_{(k)} x_j(\tau_j(kT_j)) \, T_j \, \delta(t - t_j^0 - kT_j) \\ &= \sqrt{\beta_P} \sum_{j=1}^{N} \sum_{(k)} x_j(\tau_j(0) + kT_j) \, T_j \, \delta(t - t_j^0 - kT_j) \\ &= \sqrt{\beta_P} \sum_{j=1}^{N} \sum_{(k)} x_j(kT_j) \, T_j \, \delta(t - t_j^0 - kT_j) \\ &= \sqrt{\beta_P} \sum_{j=1}^{N} \sum_{(k)} x_j(t - t_j^0) \, T_j \, \delta(t - t_j^0 - kT_j). \end{aligned} \tag{3.5}$$

Inserting for $x_j(t - t_j^0)$ the general Fourier transform

$$x_j(t - t_j^0) = \int_C dw \; \tilde{x}_j(w) e^{iw(t - t_j^0)},$$

and expanding the periodic δ-function in a Fourier series,

$$\sum_{(k)} T_j \, \delta(t - t_j^0 - kT_j) = \sum_{(m)} e^{im\omega_j(t - t_j^0)},$$

yields

$$S(t) = \sqrt{\beta_P} \sum_{j=1}^{N} \sum_{(m)} \int_C dw \; \tilde{x}_j(w) e^{-iwt_j^0} e^{-im\theta_j^0} e^{i(w + m\omega_j)t}. \tag{3.6}$$

At this point it is necessary to explain the model of the unbunched beam on which the further investigations of the collective beam motion is based.

3.4 The Model of the Unbunched Beam

3.4.1 Decomposition of the Motion into Coherent Modes

In order to describe the *collective* motion of the beam, we assume that the initial azimuths θ_j^0 of the N particles are equally distributed around the ring. Of course any other distribution

is just as possible, but with a view to analyse the macroscopic beam motion the equally spaced distribution is the simplest choice which also seems physically reasonable. Within this assumption, the betatron oscillations of the particles can be decomposed with respect to the initial azimuths,

$$\widetilde{x}_j(w)e^{-iwt_j^0} = \frac{1}{N}\sum_{l=-N/2}^{N/2-1}\widetilde{C}_l(w)e^{il\theta_j^0}. \tag{3.7}$$

Because the expansion will always be used in the limit $N \to \infty$ (see Sect. 3.4.2), an even N can be assumed without loss of generality. Performing this limit, the *symmetric* summation ensures that the index range equally extends to $\pm\infty$ whilst its center remains fixed at the origin at $l = 0$, allowing a clear identification of the conjugate expansion coefficients.

The exponential functions $e^{il\theta_j^0}$ in Eq. (3.7) form a complete mutually orthogonal set, and hence the behaviour of $\widetilde{x}_j(w)$ is completely determined by the coefficients $\widetilde{C}_l(w)$. Given that all beam particles have the same revolution frequency ω_0, the expansion leads to the normal modes of the beam and the coefficients $\widetilde{C}_l(w)$ yield the mode amplitudes [10]. In order to predict the stability of the collective beam motion it is sufficient to show that the expansion coefficients are bounded: we thus restrict further investigations to the stability of these coefficients.

3.4.2 The Limit of the Continuous Beam

If the initial azimuths of the N particles are equally distributed around the ring, adjacent particles will be separated by $\Delta\theta^0 = 2\pi/N$ so that the initial azimuth of particle j can be written as $\theta_j^0 = j \cdot \Delta\theta^0$. For a large number of particles, i.e. in the limit $N \to \infty$, the distance $\Delta\theta^0$ becomes infinitesimal small, suggesting the definition of a continuous azimuthal distribution function. The summations over the initial azimuths $\theta_j^0 = j \cdot \Delta\theta^0$ can now be replaced by integrations over a continuous variable θ^0.

By the same argument the summations with respect to the revolution frequencies ω_j can be substituted by corresponding integrations. When the frequencies of the particles are spread over an interval $\Delta\omega$, the mean frequency difference is given by $\delta\omega = \Delta\omega/N$. In the limit of large particle numbers this difference approaches zero and we can assume a continuous frequency distribution $f_0(\omega)$.

Since the particle index j marks both the initial azimuth θ_j^0 and the revolution frequency ω_j, the substitution of a summation over particles leads to a double integral over the corresponding variables θ^0 and ω. Defining a distribution function $\bar{f}(\theta^0, \omega)$ normalized to unity, we can write for any function $F(\theta^0, \omega)$

$$\sum_{j=1}^{N} F(\theta_j^0, \omega_j) \longrightarrow N\int_0^{2\pi} d\theta^0 \int_0^{\infty} d\omega\ \bar{f}(\theta^0, \omega)F(\theta^0, \omega).$$

Assuming that the revolution frequencies and initial azimuths are uncorrelated, the distribution function $\bar{f}(\theta^0, \omega)$ factorizes, and hence reads for equally distributed initial azimuths

$$\bar{f}(\theta^0, \omega) = f_\theta(\theta^0)f_0(\omega) = \frac{1}{2\pi}f_0(\omega) \qquad \text{with} \qquad \int_0^{\infty} d\omega\ f_0(\omega) = 1.$$

Under these conditions, the substitution can be written as

$$\sum_j F(\theta_j^0, \omega_j) \longrightarrow N\int \frac{d\theta^0}{2\pi}\int d\omega\ f_0(\omega)F(\theta^0, \omega). \tag{3.8}$$

We will also give a physical argument for describing the beam by continuous variables. Electronic systems have only a limited bandwidth, and thus are unable to resolve individual particles of a dense beam. By averaging over many particles, these systems smooth out the inherent discrete time structure of the signals which hence change continuously in time. Thus the measured signals in an azimuthal equally distributed beam do not reveal the discreteness of the particles, and it seems reasonable to assume a continuous beam.

The change to a continuous frequency distribution is justified by the fact that the typical frequency difference $\delta\omega$ can be resolved only after a time $t \sim 1/\,\delta\omega$ which lies far beyond the time scales considered in the following investigations. Moreover, the frequencies have to be constant over this time interval, requiring a stability of the accelerator components which technically cannot be realized. Fluctuations in the electronics (e.g. power-supply ripples or electronical noise) make it impossible to attach fixed frequencies to the particles.

3.5 The Signal Transfer from Pickup to Kicker

We now apply the continuous beam model to the signal $S(t)$ at the pickup. Starting at Eq. (3.6), we insert the expansion (3.7) and obtain

$$S(t) = \sqrt{\beta_P} \sum_j \sum_{(m)} \int dw \, \frac{1}{N} \sum_l \widetilde{C}_l(w) e^{il\theta_j^0} e^{-im\theta_j^0} e^{i(w+m\omega_j)t}.$$

For a continuous beam the particle sum can be replaced according to Eq. (3.8), yielding

$$S(t) = \sqrt{\beta_P} \sum_{(m)} \sum_{(l)} \int dw \int \frac{d\theta^0}{2\pi} \int d\omega \; f_0(\omega) \widetilde{C}_l(w) e^{i(l-m)\theta^0} e^{i(w+m\omega)t}.$$

Since

$$\int \frac{d\theta^0}{2\pi} \, e^{i(l-m)\theta^0} = \delta_{ml},$$

it follows that

$$S(t) = \sqrt{\beta_P} \sum_{(m)} \int dw \int d\omega \; f_0(\omega) \widetilde{C}_m(w) e^{i(w+m\omega)t}.$$

The general Fourier transform of $S(t)$ reads

$$\widetilde{S}(w) = \frac{\sqrt{\beta_P}}{2\pi} \sum_{(m)} \int d\omega \; f_0(\omega) \widetilde{C}_m(w - m\omega). \tag{3.9}$$

Assuming a linear impulse response $G(t)$ which models the complete cooling system including the pickup and kicker, we write the signal transfer from the pickup to kicker as (see Sect. 2.2)

$$G\,[S]\,(t) = \int_0^\infty dt' \; G(t - t') S(t').$$

Using Eq. (A.5), the Fourier transformation yields

$$\widetilde{G\,[S]}(w) = \widetilde{G}(w) \widetilde{S}(w),$$

so that after the inverse transformation the transferred signal is given by

$$G\,[S]\,(t) = \int dw \; e^{iwt} \widetilde{G}(w) \widetilde{S}(w). \tag{3.10}$$

Eqs. (3.9) and (3.10) determine the force acting on the beam at the kicker.

3.6 The Self-Consistent Solution to the Beam Motion

3.6.1 The Motion of a Single Particle

The Equation Of Motion

In the notation of Section 3.2 the equation of motion of the particle j reads

$$\frac{d^2}{d\tau_j^2}\, x_j\Big(\tau_j(t-t_j^0)\Big) + \Omega_j^2\, x_j\Big(\tau_j(t-t_j^0)\Big) = \Omega_j^2 \beta_K^{3/2}\Big(F_j(t)+g_j(t)\Big) \tag{3.11}$$

where β_K is the value of the beta function at the kicker. $F_j(t)$ and $g_j(t)$ denote the feedback force and an external excitation respectively. Because the particle passes the localized kicker only at discrete times, the forces $F_j(t)$ and $g_j(t)$ are sampled quantities.

Sampled Forces at the Kicker

Let θ^K denote the position of the kicker, and $\theta^{PK} = \theta^K - \theta^P$ its azimuthal distance to the pickup (see Fig. 3-1). The particle samples the forces at the kicker at times $t_k = t_j^0 + t_j^{PK} + kT_j$ with $k = 0, \pm 1, \pm 2, \ldots$ where t_j^{PK} is the transit time of the particle between pickup and kicker, i.e. $\theta^{PK} = \omega_j t_j^{PK}$. Hence the sampled forces $F_j(t)$ and $g_j(t)$ can be written as

$$\begin{aligned} F_j(t) &= \sum_{(k)} T_j\, \delta(t - t_j^0 - t_j^{PK} - kT_j)\, G\,[S]\,(t) \\ &= \sum_{(k)} T_j\, \delta(t - t_j^0 - t_j^{PK} - kT_j)\, G\,[S]\,(t_j^0 + t_j^{PK} + kT_j) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} g_j(t) &= \sum_{(k)} T_j\, \delta(t - t_j^0 - t_j^{PK} - kT_j)\, g(t) \\ &= \sum_{(k)} T_j\, \delta(t - t_j^0 - t_j^{PK} - kT_j)\, g(t_j^0 + t_j^{PK} + kT_j). \end{aligned} \tag{3.13}$$

$G\,[S]\,(t)$ is given by Eq. (3.10) and $g(t)$ describes the continuous external excitation.

Transformation of the Equation of Motion

By substituting $\bar{\tau}_j = \tau_j(t - t_j^0)$ in Eq. (3.11) and observing that $d\bar{\tau}_j = d\tau_j$, we obtain

$$\frac{d^2}{d\bar{\tau}_j^2}\, x_j(\bar{\tau}_j) + \Omega_j^2\, x_j(\bar{\tau}_j) = \Omega_j^2 \beta_K^{3/2}\Big(F_j(t) + g_j(t)\Big). \tag{3.14}$$

The general Fourier transform with respect to the quasi-time $\bar{\tau}_j$ is defined by

$$\widetilde{f}(w) = \frac{1}{2\pi}\int_0^\infty d\bar{\tau}_j\; f(\bar{\tau}_j) e^{-iw\bar{\tau}_j}$$

and thus the transformed equation of motion (3.14) reads

$$(-w^2 + \Omega_j^2)\, \widetilde{x}_j(w) = \Omega_j^2 \beta_K^{3/2}\left(\widetilde{F}_j(w) + \widetilde{g}_j(w)\right), \tag{3.15}$$

where

$$\widetilde{F}_j(w) = \int d\bar{\tau}_j \; F_j(t) e^{-iw\bar{\tau}_j} \tag{3.16}$$

and

$$\widetilde{g}_j(w) = \int d\bar{\tau}_j \; g_j(t) e^{-iw\bar{\tau}_j}. \tag{3.17}$$

Transformation of the Feedback Force

Now the Fourier transform of the feedback force $\widetilde{F}_j(w)$ is further evaluated. Starting with Eq. (3.16),

$$\begin{aligned} \widetilde{F}_j(w) &= \int d\bar{\tau}_j \; F_j(t) e^{-iw\bar{\tau}_j} \\ &= \int d\tau_j \; F_j(t) e^{-iw\tau_j(t-t_j^0)}, \end{aligned}$$

we replace the integration over τ_j by an integration over t by using the relation (3.3) and obtain by Eq. (3.12)

$$\widetilde{F}_j(w) = \frac{\omega_j}{\Omega_j} \frac{R}{\beta_K} \int dt \sum_{(k)} T_j \, \delta(t - t_j^0 - t_j^{PK} - kT_j) \, G\,[S]\,(t_j^0 + t_j^{PK} + kT_j) e^{-iw\tau_j(t-t_j^0)}.$$

Using of the periodicity (3.2), we find for the quasi-time in the exponent after integration

$$\tau_j(t_j^0 + t_j^{PK} + kT_j - t_j^0) = \tau_j(t_j^{PK} + kT_j) = \tau_j^{PK} + kT_j$$

where $\tau_j^{PK} = \tau_j(t_j^{PK})$. Together with Eq. (3.10), it follows

$$\begin{aligned} \widetilde{F}_j(w) &= \frac{1}{\Omega_j} \frac{2\pi R}{\beta_K} \sum_{(k)} e^{-iwkT_j} e^{-iw\tau_j^{PK}} \int dw' \; \widetilde{G}(w') \widetilde{S}(w') e^{iw'(t_j^0 + t_j^{PK} + kT_j)} \\ &= e^{-iw\tau_j^{PK}} \frac{1}{\Omega_j} \frac{2\pi R}{\beta_K} \int dw' \; \widetilde{G}(w') \widetilde{S}(w') e^{iw't_j^{PK}} e^{iw't_j^0} \sum_{(k)} e^{i(w'-w)kT_j}. \end{aligned} \tag{3.18}$$

Since

$$\sum_{(k)} e^{i(w'-w)kT_j} = \omega_j \sum_{(m)} \delta(w' - w - m\omega_j),$$

this reduces to

$$\widetilde{F}_j(w) = e^{-iw\tau_j^{PK}} \frac{\omega_j}{\Omega_j} \frac{2\pi R}{\beta_K} \sum_{(m)} \widetilde{G}(w + m\omega_j) \widetilde{S}(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} e^{i(w+m\omega_j)t_j^0}. \tag{3.19}$$

Finally, we define a periodic function $\widehat{F}_j(w)$ by

$$\widehat{F}_j(w) = e^{iw\tau_j^{PK}} \widetilde{F}_j(w), \tag{3.20}$$

which has the period ω_j because $\widehat{F}_j(w + l\omega_j) = \widehat{F}_j(w)$ for any integer l (see Sect. 2.5).

Transformation of the External Excitation

Proceeding with Eqs. (3.13) and (3.17) in the same way, we obtain an expression similar to the result (3.18)

$$\begin{aligned}\widetilde{g}_j(w) &= \frac{1}{\Omega_j}\frac{2\pi R}{\beta_K}\sum_{(k)} e^{-iwkT_j} e^{-iw\tau_j^{PK}} \int dw'\, \widetilde{g}(w') e^{iw'(t_j^0 + t_j^{PK} + kT_j)} \\ &= e^{-iw\tau_j^{PK}} \frac{1}{\Omega_j}\frac{2\pi R}{\beta_K} \int dw'\, \widetilde{g}(w') e^{iw't_j^{PK}} e^{iw't_j^0} \sum_{(k)} e^{i(w'-w)kT_j}\end{aligned}$$

so that

$$\widetilde{g}_j(w) = e^{-iw\tau_j^{PK}} \frac{\omega_j}{\Omega_j}\frac{2\pi R}{\beta_K} \sum_{(m)} \widetilde{g}(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} e^{i(w+m\omega_j)t_j^0}. \tag{3.21}$$

Analogously, we define the periodic function $\widehat{g}_j(w)$ by

$$\widehat{g}_j(w) = e^{iw\tau_j^{PK}} \widetilde{g}_j(w) \tag{3.22}$$

with the property $\widehat{g}_j(w + l\omega_j) = \widehat{g}_j(w)$.

3.6.2 The Transition to the Continuous Beam

The single particle results which have been obtained thus far are now employed to derive the self-consistent solution to the collective beam motion in the framework of the continuous beam model established in Section 3.4. At this time, however, we will only present the important results and refer the detailed calculations to Appendix C.

Before proceeding, it is necessary to make some approximations which are of use in the derivations.

(i) We assume zero chromaticity, $\xi = 0$. Then all particles have the same tune Q and their betatron frequencies are given by $\Omega_j = Q\omega_j$.[1]

(ii) If φ^{PK} denotes the betatron phase advance of the particles between pickup and kicker, we can write $\tau_j^{PK} = \varphi^{PK}/Q\omega_j$.

The derivation starts at the transformed equation of motion (3.15) which by Eqs. (3.20) and (3.22) can be written as

$$\left(-w^2 + \Omega_j^2\right) \widetilde{x}_j(w) = \Omega_j^2 \beta_K^{3/2} e^{-iw\tau_j^{PK}} \left(\widehat{F}_j(w) + \widehat{g}_j(w)\right). \tag{3.23}$$

From this equation we can derive the self-consistent solution to the beam signal $S(t)$ at the pickup. According to Eq. (C.8), the general Fourier transform $\widetilde{S}(w)$ of $S(t)$ is

$$\widetilde{S}(w) = \frac{\widetilde{R}(w)\widetilde{g}(w)}{1 - \widetilde{R}(w)\widetilde{G}(w)} \tag{3.24}$$

[1] This means no essential restriction because a frequency-dependent tune could easily be taken into account in the further formalism by the substitution $Q \to Q(\omega)$. However, this would produce unnecessary complex equations which basically do not provide any new physical insight.

with the definition

$$\widetilde{R}(w) = \kappa N \int_0^\infty d\omega \; f_0(\omega)\, \omega^2 \sum_{(m)} \frac{e^{-i(w+m\omega)\varphi^{PK}/Q\omega}}{(Q\omega)^2 - (w+m\omega)^2}\, e^{iw\theta^{PK}/\omega} \tag{3.25}$$

where $\kappa = RQ\sqrt{\beta_P\,\beta_K}$.

Using Eq. (3.24), we can immediately derive the force acting on the beam, and a determination of the coefficients $\widetilde{C}_l(w)$ becomes possible. This step is also carried out in Appendix C, yielding the result (C.13),

$$\widetilde{C}_l(w) = \bar{\kappa} N \int_0^\infty d\omega \; f_0(\omega)\, \omega^2 \, \frac{e^{-iw\varphi^{PK}/Q\omega}}{(Q\omega)^2 - w^2}\, \frac{\widetilde{g}(w+l\omega)e^{i(w+l\omega)\theta^{PK}/\omega}}{1 - \widetilde{G}(w+l\omega)\widetilde{R}(w+l\omega)} \tag{3.26}$$

with $\bar{\kappa} = 2\pi RQ\sqrt{\beta_K}$. Now the stage is set for an investigation of the stability behaviour of the beam.

3.7 Stability of the Beam

The results of the preceding section provide the basis for a stability analysis of the coefficient $C_l(t)$. The time behaviour and so the stability of each $C_l(t)$ is completely determined by the singularities of the Fourier transform $\widetilde{C}_l(w)$ (see Sect. 2.3), and hence we must find the zeros in the denominator of Eq. (3.26). At first glance, we would expect that the singularities are determined by the two conditions

$$(1) \qquad (Q\omega')^2 - w_0^2 = 0 \tag{3.27}$$

$$(2) \qquad \widetilde{G}(w_l + l\omega')\, \widetilde{R}(w_l + l\omega') = 1. \tag{3.28}$$

However within the bandwidth of the system, i.e. for $\widetilde{G}(w_0 + l\omega') \neq 0$, the zeros w_0 of Eq. (3.27) do not result in singularities, since for each term $(Q\omega)^2 - w^2$ in Eq. (3.26) there exists its reciprocal $[(Q\omega)^2 - w^2]^{-1}$ in the function $\widetilde{R}(w + l\omega)$ which cancels the apparent singularity at $w^2 = (Q\omega)^2$. Only if the frequencies $w_0 + l\omega'$ lie outside the bandwidth of $\widetilde{G}(w)$, i.e. for $\widetilde{G}(w_0 + l\omega') = 0$, the zeros w_0 will be important for the time behaviour of the coefficients $C_l(t)$. Because these coefficients are not influenced by the cooling system, they cannot become unstable by an interaction via the feedback loop and therefore need not to be considered for our purpose.

Predictions about the stability of the collective beam motion in a cooling system can only be obtained from the frequencies w_l which satisfy the second condition (3.28), and thus depend on the gain $\widetilde{G}(w)$ of the cooling system. The *critical gain* $\widetilde{G}_{crit}(w)$ is defined as that value at which the corresponding frequencies w_l describe the onset of unstable collective beam motion, and therefore determines the stability boundaries of the system. There is a continuous functional dependence of the gain $\widetilde{G}(w)$ from the roots w_l so that with $\gamma_l = \mathsf{Im}\, w_l$:

$$\gamma_l \longrightarrow 0 \quad \Longleftrightarrow \quad \widetilde{G}(w_l) \longrightarrow \widetilde{G}_{crit}(w_l).$$

To determine $\widetilde{G}_{crit}(w_l)$, we first assume an initially stable solution w_l of Eq. (3.28) with $\gamma_l > 0$, and then we let the imaginary part γ_l tend to zero, yielding the defining equation for the critical gain:

$$\lim_{\gamma_l \to 0^+} \left[1 - \widetilde{G}(w_l + l\omega')\, \widetilde{R}(w_l + l\omega')\right] = 0,$$

and with the definition $\Omega_l = \mathsf{Re}\, w_l + l\omega'$ it follows

$$1 = \widetilde{G}_{crit}(\Omega_l) \lim_{\gamma_l \to 0^+} \widetilde{R}(\Omega_l + i\gamma_l). \tag{3.29}$$

The evaluation of the limit is performed in Appendix D, leading to the stability criterion (D.5),

$$\begin{aligned} \frac{Q}{\kappa N |\widetilde{G}_{crit}(\Omega_l)|} &= i\pi \sum_{(m)} \frac{1}{|m+Q|} f\!\left(\frac{\Omega_l}{m+Q}\right) \frac{\Omega_l}{m+Q} e^{i\bar{\Phi}^{\delta}_m(\Omega_l)} + \\ & \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\, f(\omega)\, \omega \frac{e^{i\bar{\Phi}^{\mathcal{P}}_m(\Omega_l,\omega)}}{(m+Q)\omega + \Omega_l}. \end{aligned} \tag{3.30}$$

with the distribution function $f(\omega)$ being redefined by Eq. (D.1). The phases $\bar{\Phi}^{\delta}_m(\Omega_l)$ and $\bar{\Phi}^{\mathcal{P}}_m(\Omega_l,\omega)$ are given by the relations (D.6) and (D.7) respectively.

When applied to cooling systems, Eq. (3.30) can be further reduced, since

- Cooling systems are usually set up such that the phase $\psi(w)$ of the feedback gain can be described by a pure delay τ, so we can write

$$\widetilde{G}(w) = |\widetilde{G}(w)| e^{-iw\tau}.$$

- The signal delay τ in the electronic components is adjusted to match the transit times of the particles between pickup and kicker. As a result of their distribution in revolution frequencies the particles have different transit times so that this coincidence cannot be accomplished for all particles simultaneously. The delay is generally adapted to the center frequency ω_0 of the distribution, thus accommodating the most particles. In this case we obtain

$$\tau = t_0^{PK} = \theta^{PK}/\omega_0.$$

- The pickup detects the transverse displacements of the particles whereas the kicker corrects the angle corresponding to the measured errors, requiring the proper betatron phase advance φ^{PK} between pickup and kicker. Ideally, the azimuthal distance θ^{PK} from pickup to kicker is chosen such that the betatron phase advance between them is just

$$\varphi^{PK} = \frac{\pi}{2} + 2\pi n, \qquad n = 0, 1, 2, \ldots$$

- Assuming only small real frequency shifts due to feedback interaction, the transverse frequencies can be written as

$$\mathsf{Re}\, w_l \approx Q\omega' \qquad \text{and} \qquad \Omega_l \approx (l+Q)\omega'.$$

  From Eq. (3.26) it can be seen that $\omega' > 0$ and hence $\sigma_{\Omega_l} = \sigma_{\underline{l}}$.[2]

With these assumptions, Eq. (3.30) can be further evaluated (see App. D), and we obtain the following stability criteria,

$$\frac{Q}{\kappa N |\widetilde{G}_{crit}(\Omega_l)|} = \pi \sum_{(m)} \frac{\cos \Phi^{\delta}_m(\Omega_l)}{|m+Q|} \frac{|\Omega_l|}{|m+Q|} f\!\left(\frac{\Omega_l}{m+Q}\right) + \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\, f(\omega)\, |\omega| \frac{\sin \Phi^{\mathcal{P}}_m(\Omega_l,\omega)}{(m+Q)\omega + \Omega_l} \tag{3.31}$$

[2]The quantities σ_{Ω_l}, $\sigma_{\underline{l}}$ and σ_ω are defined in Appendix D.

and

$$\pi \sum_{(m)} \frac{\sin \Phi^{\delta}_{m}(\Omega_l)}{|m+Q|} \frac{|\Omega_l|}{|m+Q|}\, f\!\left(\frac{\Omega_l}{m+Q}\right) = \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\, \frac{\cos \Phi^{\mathcal{P}}_{m}(\Omega_l,\omega)}{(m+Q)\omega+\Omega_l} \tag{3.32}$$

where the phases are given by

$$\Phi^{\delta}_{m}(\Omega_l) \approx \sigma_{\underline{l}} \Big[|l+Q| - |m+Q|\Big]\theta^{PK} + (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0} \tag{3.33}$$

and

$$\Phi^{\mathcal{P}}_{m}(\Omega_l,\omega) \approx \Big[(l+Q) + \sigma_{\omega}\,(m+Q)\Big]\frac{\varphi^{PK}}{Q} + (l+Q)\frac{\Delta\varphi}{Q}\Delta'(\omega) + (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}. \tag{3.34}$$

These equations which allow for overlapping frequency bands in the beam spectrum establish the stability boundaries of the expansion coefficients $\widetilde{C}_l(w)$ over the entire frequency range. Eq. (3.31) defines the maximum stable gain of the cooling system: if $|\widetilde{G}(w)|$ exceeds the critical value $|\widetilde{G}_{crit}(w)|$ the beam will become unstable owing to the feedback interaction. On the other hand, Eq. (3.32) determines the real frequencies $\Omega_l = \mathsf{Re}\, w_l + l\omega'$ at which the beam supports the propagation of collective modes due to the critical gain of the cooling system.

The Special Case of Non-Overlapping Bands

The results obtained above include also the special case of non-overlapping frequency bands. In this case particles can only interact via frequencies lying within the same band, and thus the summation over the bands reduces to the single term $m = l$.
Using $\Omega_l = (l+Q)\omega'$ and $\omega' > 0$, the Eqs. (3.31) and (3.32) now reads

$$\frac{Q}{\kappa N |\widetilde{G}_{crit}(\Omega_l)|} = \pi \frac{\cos \Phi^{\delta}_{l}(\Omega_l)}{|l+Q|}\,\omega'\, f(\omega') + \frac{1}{l+Q}\,\mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\,\frac{\sin \Phi^{\mathcal{P}}_{l}(\Omega_l,\omega)}{\omega+\omega'} \tag{3.35}$$

and

$$\pi \frac{\sin \Phi^{\delta}_{l}(\Omega_l)}{|l+Q|}\omega'\; f(\omega') = \frac{1}{l+Q}\,\mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\,\frac{\cos \Phi^{\mathcal{P}}_{l}(\Omega_l,\omega)}{\omega+\omega'}. \tag{3.36}$$

The phases (3.33) and (3.34) become

$$\Phi^{\delta}_{l}(\Omega_l) = \sigma_{\underline{l}} \Big[|l+Q| - |l+Q|\Big]\theta^{PK} + (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0} = (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}$$

and

$$\Phi^{\mathcal{P}}_{l}(\Omega_l,\omega) = \Big[(l+Q) + \sigma_{\omega}\,(l+Q)\Big]\frac{\varphi^{PK}}{Q} + (l+Q)\frac{\Delta\varphi}{Q}\Delta'(\omega) + (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}$$

respectively. Significant contributions from the principal value integrals in the Eqs. (3.35) and (3.36) only arise in the proximity of the pole, i.e. for $\omega \approx -\omega'$. Since $\omega' > 0$ (see Eq. (3.26)) and thus $\sigma_\omega = -1$, we obtain

$$\Phi^{\mathcal{P}}_{l}(\Omega_l,\omega) = (l+Q)\frac{\Delta\varphi}{Q}\Delta'(\omega) + (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}.$$

These results are consistent with the findings in [9] which have been obtained by considering only non-overlapping frequency bands. The more general theory (including band overlap) gives us additional information about the phases $\Phi^{\delta}_{l}(\Omega_l)$ and $\Phi^{\mathcal{P}}_{l}(\Omega_l,\omega)$ which in [9] have been assumed to be zero.

3.8 Comparison with the Feedback Theory

Finally, we will show that the result (3.26) for the mode expansion coefficients is consistent with the predictions of the multi-bunch feedback theory. For that purpose we assume a mono-energetic beam, and thus the same revolution frequency ω_0 for all particles, which later will permit to identify the beam particles with bunches. Then the distribution function in revolution frequencies reads $f_0(\omega) = \delta(\omega - \omega_0)$, and Eq. (3.26) reduces to

$$\widetilde{C}_l(w) = \bar{\kappa}\omega_0^2 N \frac{e^{-iw\tau^{PK}}}{(Q\omega_0)^2 - w^2} \frac{\widetilde{g}(w + l\omega_0)e^{i(w+l\omega)t^{PK}}}{1 - \widetilde{G}(w + l\omega_0)\widetilde{R}(w + l\omega_0)}$$

where $\tau^{PK} = \varphi^{PK}/Q\omega_0$ and $t^{PK} = \theta^{PK}/\omega_0$ are now the same for all particles. Furthermore, Eq. (3.25) becomes

$$\widetilde{R}(w) = \kappa\omega_0^2 N e^{iwt^{PK}} \sum_{(m)} \frac{e^{-i(w+m\omega_0)\tau^{PK}}}{(Q\omega_0)^2 - (w + m\omega_0)^2}.$$

Defining the ω_0-periodic function

$$\widehat{R}(w) = \sum_{(m)} \frac{e^{-i(w+m\omega_0)\tau^{PK}}}{(Q\omega_0)^2 - (w + m\omega_0)^2},$$

and writing the feedback gain and the external mode excitation as

$$\widetilde{G}_C(w) = \widetilde{G}(w)e^{iwt^{PK}} \tag{3.37}$$

and

$$\frac{1}{N}\widetilde{F}_l(w) = \widetilde{g}(w + l\omega_0)e^{i(w+l\omega_0)t^{PK}}$$

respectively, we obtain

$$\widetilde{C}_l(w) = \frac{e^{-iw\tau^{PK}}}{(Q\omega_0)^2 - w^2} \frac{\bar{\kappa}\omega_0^2 \widetilde{F}_l(w)}{1 - \Upsilon\widetilde{G}_C(w + l\omega_0)\widehat{R}(w)} \tag{3.38}$$

where $\Upsilon = \kappa\omega_0^2 N$. The corresponding result from the multi-bunch feedback theory for the r-th multi-bunch mode is given by [10]

$$\widetilde{C}_r(w) = \sum_{(m)} \frac{e^{-i(w+m\omega_0)\tau^{PK}}}{(Q\omega_0)^2 - (w + m\omega_0)^2} \frac{\bar{\kappa}\omega_0^2 \widetilde{F}_r(w)}{1 - \Upsilon\widetilde{G}_{CN}(w + r\omega_0)\widehat{R}(w)} \tag{3.39}$$

with the definition

$$\widetilde{G}_{CN}(w) = \sum_{(l)} \widetilde{G}(w + lN\omega_0)e^{i(w+lN\omega_0)t^{PK}},$$

in which N denotes the number of bunches. $\widetilde{G}_{CN}(w)$ is a periodic function with period $N\omega_0$.

We now discuss the differences between the Eqs. (3.38) and (3.39). To that end we identify the bunches in the feedback theory with the individual particles of the unbunched beam.

The period $N\omega_0 = 2\pi N f_0$ of $\widetilde{G}_{CN}(w)$ equals (except for a factor 2π) the bunch frequency f_B, i.e. the frequency with which succeeding bunches appear at fixed locations. If T_B denotes

the time difference between two adjacent bunches and T_0 the revolution time, it immediately follows that $f_B = 1/T_B = 1/(T_0/N) = N f_0$. The limit of the continuous unbunched beam in which the distance of adjacent particles becomes zero (see Sect. 3.4.2) corresponds in the bunched beam to $T_B \to 0$ or $f_B \to \infty$ so that $\widetilde{G}_{CN}(w)$ is no longer periodic. In this limit $\widetilde{G}_{CN}(w)$ turns into the function $\widetilde{G}_C(w)$ which has been defined for the unbunched beam (see Eq. (3.37)).

The second difference between the Eqs. (3.38) and (3.39) shows up in the first term on the right hand side of each equation. The expression for the multi-bunch modes contains an additional summation over all revolution harmonics $m\omega_0$ which arises from the discrete time structure of the signals and expresses the *periodic sampling* of the force at the kicker by the bunches (see Sect. 2.5). The continuous beam, on the other hand, is found at the kicker at any time and therefore does *not sample* the force at discrete times, and hence the summation over the revolution harmonics does not occur in Eq. (3.39). In this respect the continuous beam corresponds to a bunched beam subjected to a force which is distributed over the whole ring. Since in this case the force would act permanently on the bunches, the summation in Eq. (3.39) would likewise disappear.

If, according to these comments, we replace in Eq. (3.39)

$$\widetilde{G}_{CN}(w) \longrightarrow \widetilde{G}_C(w)$$

and disregard the sampling of the force so that

$$\sum_{(m)} \frac{e^{-i(w+m\omega_0)\tau^{PK}}}{(Q\omega_0)^2 - (w + m\omega_0)^2} \longrightarrow \frac{e^{-iw\tau^{PK}}}{(Q\omega_0)^2 - w^2},$$

the result of the multi-bunch feedback theory will yield expression (3.38) which has been obtained for the unbunched beam.

Chapter 4

Stochastic Cooling of Unbunched Beams

4.1 Overview

In this chapter we will discuss transverse stochastic cooling of an unbunched beam in case of a linear cooling interaction. The basic concepts of stochastic cooling are presented on a qualitative level, omitting the detailed derivations of the fundamental equations for the most part. Only the calculations which are typical of the specific cooling process considered here, are explicitly shown in the Appendices E and F.

Compared with the instabilities studied in the previous chapter stochastic cooling can be regarded as a slow process. Hence it is convenient to investigate the cooling in the time domain since then averaging over fast changing variables can easily be performed. Here, we will describe the cooling process by means of a Fokker-Planck equation for the phase-space density in the transverse action variable. Unlike the sample picture of stochastic cooling [6] where the properties of the cooling process have to be introduced explicitly on the basis of empirical arguments, these effects follow automatically in the Fokker-Planck approach due to the more rigorous mathematical treatment of the particle dynamics. In the results we will recognize the *mixing* which measures the decay of inter-particle correlations as well as the *optimal betatron phase advance* of the particles between the pickup and kicker. Furthermore the reason will become obvious why stochastic cooling systems take a *shortcut through the ring*. The results of the Fokker-Planck equation allow a profound insight into the physical origin of these effects and can also provide quantitative predictions for the parameters which determine the performance of the cooling system. This will be discussed in detail in the Sections 4.5.3 and 4.5.4.

The Fokker-Planck equation represents a statistical description of the cooling process, assuming independent particles with random phases. In Section 4.5.2 this assumption will be motivated by means of physical arguments. Including the collective motion of the particles into this formulation poses considerable mathematical difficulties, and hence further assumptions have to be made. The cooling model underlying the Fokker-Planck equation neglects the collective effects (see Sect. 4.5.1), and therefore can only be applied after the beam stability has explicitly been proven. The parameters for which the Fokker-Planck equation predicts the most efficient cooling operation are compared to the stability criteria of the coherent beam modes in order to verify that these predictions are compatible with the stability boundaries of the beam. Initially, we will review the fundamental physical principles of stochastic cooling.

4.2 The Phase-Space Fluctuations

The beam model used in the previous chapter is capable of describing collective effects for which not the single particle behaviour, but the common motion of all particles is important. The model considers the phase-space volume occupied by the beam as a whole, thus providing only information about macroscopic beam quantities averaged over all particles. The macroscopic state of the beam is completely characterized by a smooth mean phase-space density which no longer contains information about individual particles.

The beam cooling aims at increasing the phase-space density by concentrating the beam particles in a phase-space volume as small as possible. To manipulate the individual particle motions, the cooling interaction needs information about the *internal* phase-space structure which to a certain degree requires the knowledge of the phase-space coordinates of single particles. Stochastic cooling finally depends on how fast and how precisely details of the phase space can be resolved in order to obtain the necessary information.

The graininess of the beam, i.e. the discrete nature and finite number of its particles, makes these information accessible to a pickup. We will illustrate this fact with a beam of N independent particles represented by N individual points in the phase space (see Fig. 4-1). To define a density, we subdivide the phase space into small volumes in which the phase-space coordinates do not change noticeably, and identify the local phase-space density with the number of particles in such a subvolume ΔV. The density defined in this way is not a continuous function, but fluctuates from subvolume to subvolume according to the number of particles each volume contains. The smooth mean phase-space density then is the ensemble average over all possible particle configurations in the subvolumes.

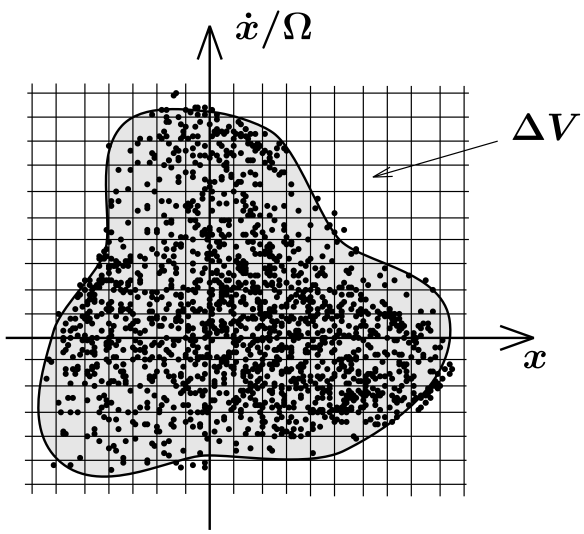

Figure 4-1: *Fluctuations of the phase-space density due to the discreteness of the beam particles.*

The actual phase-space density $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$ of the beam depends on the instantaneous phase-space coordinates of the particles and fluctuates around the smooth mean density $f(\mathbf{x}, \dot{\mathbf{x}}, t)$. Writing these fluctuations as

$$\delta f(\mathbf{x}, \dot{\mathbf{x}}, t) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) - \langle \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) \rangle = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) - f(\mathbf{x}, \dot{\mathbf{x}}, t),$$

we can describe the macroscopic and microscopic beam properties separately by means of the corresponding distribution functions, $f(\mathbf{x}, \dot{\mathbf{x}}, t)$ and $\delta f(\mathbf{x}, \dot{\mathbf{x}}, t)$. Since only the fluctuations

$\delta f(\mathbf{x}, \dot{\mathbf{x}}, t)$ contain the instantaneous phase-space coordinates of individual particles, solely they can provide the wanted information about the internal structure of the phase space.

4.3 The Fluctuation Spectrum

The fluctuations in the phase-space density are observable in the measured beam signal $S(t)$ where they lead to fast variations $\delta S(t)$ around the mean signal. The phase-space fluctuations are determined by the individual particle motions, and thus depend on the initial conditions of the particles. Apart from very special cases, the initial state of the beam is unknown, allowing only a statistical description of the fluctuations which turns them into stochastic quantities.

The time behaviour of the fluctuations follows from the autocorrelation-function,

$$C_{\delta S}(t, t') := \langle \delta S(t)\, \delta S(t') \rangle,$$

which measures the correlation time τ_{corr} of the signal fluctuations. The correlation time is given by the time interval $t' - t$ within which the autocorrelation-function of the measured signals has non-zero values. In other words, the fluctuations will be statistically independent from each other if they are separated in time by more than the correlation time τ_{corr}:

$$\langle \delta S(t)\, \delta S(t') \rangle \approx 0 \qquad \text{for} \quad \tau_{corr} < |t' - t|.$$

For unbunched beams, the autocorrelation-function only depends on the difference $\tau = t' - t$, and the Fourier transformation yields the power spectrum of the fluctuations which often is referred to as Schottky spectrum,

$$P_{\delta S}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\; e^{-i\omega\tau}\, \langle \delta S(t)\, \delta S(t+\tau) \rangle.$$

At this place, we omit a detailed description of the Schottky spectra, and summarize only the properties important for the following discussion. General information about the Schottky beam-spectra can be found e.g. in [23].

The Schottky spectrum mirrors the revolution frequency distribution of the particles so that the results of Section 2.6 also apply to the Schottky spectrum. One obtains a spectrum of bands in which the width of the bands increases at higher frequencies, resulting in a band overlap beyond a certain frequency. This corresponds to a finite correlation time in the order of the reciprocal width of the bands, and thus decreases at higher frequencies. We will see later that a finite correlation time, and with it a non-zero width of the frequency distribution, are essential for the stochastic cooling operation.

4.4 Particle Dynamics in the Cooling System

To describe the stochastic beam cooling, we again start at the single particle equation of motion, but this time do not discard the discreteness of the particles. Excluding any external excitation, it follows from Eqs. (3.14) and (3.19) for the particle j

$$\frac{d^2}{d\bar{\tau}_j^2}\, x_j(\bar{\tau}_j) + \Omega_j^2\, x_j(\bar{\tau}_j) = \Omega_j^2 \beta_k^{3/2} \int dw\; e^{iw\bar{\tau}_j}\, \widetilde{F}_j(w) \tag{4.1}$$

where

$$\widetilde{F}_j(w) = e^{-iw\tau_j^{PK}} \frac{\omega_j}{\Omega_j} \frac{2\pi R}{\beta_k} \sum_{(m)} \widetilde{G}(w + m\omega_j)\widetilde{S}(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} e^{i(w+m\omega_j)t_j^0}. \tag{4.2}$$

Using Eq. (3.6), the Fourier transform of the signal at the pickup reads

$$\widetilde{S}(w) = \frac{\sqrt{\beta_P}}{2\pi} \sum_{j'=1}^{N} \sum_{(l)} \widetilde{x}_{j'}(w - l\omega_{j'}) e^{-iwt_{j'}^0}. \tag{4.3}$$

We now regard the particle j as a test-particle treated separately from the rest of the beam which consists of all the remaining particles $j' \neq j$. For this purpose, we split the force on the right-hand side of Eq. (4.1) into a part arising from the test-particle, and the contributions of all other particles. In Eq. (4.3) only the term for $j' = j$ enters into the self-interaction, and together with Eq. (4.2) we find

$$F_j^S(\bar{\tau}_j) = \kappa\omega_j^2 \int dw\; e^{iw\bar{\tau}_j} e^{-iw\tau_j^{PK}} \sum_{(m)} \widetilde{G}(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} \sum_{(l)} \widetilde{x}_j(w + l\omega_j) \tag{4.4}$$

where $\kappa = RQ\sqrt{\beta_P \beta_K}$. The contributions of the other particles are given by

$$F_j^R(\bar{\tau}_j) = \kappa\omega_j^2 \int dw\; e^{iw\bar{\tau}_j} e^{-iw\tau_j^{PK}} \sum_{(m)} \widetilde{G}(w + m\omega_j)\, \widetilde{S}_j(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} e^{i(w+m\omega_j)t_j^0} \tag{4.5}$$

with

$$\widetilde{S}_j(w) := \sum_{j' \neq j} \sum_{(l)} \widetilde{x}_{j'}(w + l\omega_{j'}) e^{-iwt_{j'}^0}.$$

Writing the cooling force in this way, it becomes more apparent that the self-interaction of the test-particle does not depend on the time $t_j^0 = \theta_j^0/\omega_j$ of the first passage of the test-particle at the pickup. The self-interaction has a fixed phase with respect to the test-particle, and thus gives a coherent contribution to the cooling interaction over the entire frequency range. On the other hand, the signal $\widetilde{S}_j(w)$ entering into the force $F_j^R(\bar{\tau}_j)$ includes the initial conditions of the particles, and shows fast variations in time, as described above. Because this fluctuations occur around the smooth macroscopic beam signal, it appears obvious to divide the force $F_j^R(\bar{\tau}_j)$ further into a collective and a fluctuation part:

$$F_j^R(\bar{\tau}_j) = \langle F_j^R(\bar{\tau}_j) \rangle + \delta F_j^R(\bar{\tau}_j).$$

The averaging process $\langle \cdot \rangle$ is performed over the phases and azimuths of the rest beam particles with $j' \neq j$. The collective force $\langle F_j^R(\bar{\tau}_j) \rangle$ results from the common motion of the particles and is completely determined by the smooth mean phase-space density. The fluctuation force $\delta F_j^R(\bar{\tau}_j)$ can be regarded as a purely statistical quantity, providing a stochastic part to the interaction.

The equation of motion of the test-particle describes a stochastic differential equation which we write as

$$\frac{d^2}{d\bar{\tau}_j^2} x_j(\bar{\tau}_j) + \Omega_j^2\, x_j(\bar{\tau}_j) = F_j^S(\bar{\tau}_j) + \langle F_j^R(\bar{\tau}_j) \rangle + \delta F_j^R(\bar{\tau}_j). \tag{4.6}$$

The stochastic force $\delta F_j^R(\bar{\tau}_j)$ in the interaction renders determined predictions of the particle motions impossible so that only the statistical properties of the kinetic quantities, obtained by averaging over the probability densities of the particles in the phase space, are relevant to the cooling description. Since the probability densities of the particles and the phase-space density of the beam are connected [14], the time evolution of the latter fully characterizes the dynamics of the cooling process.

4.5 The Time Evolution of the Phase-Space Density

4.5.1 The Model of the Cooling Interaction

Describing the cooling process in terms of the smooth mean phase-space density allows, in principle, the calculation of all statistical moments of the dynamic beam quantities. With some assumptions about the forces on the right-hand side of Eq. (4.6), one can derive a Fokker-Planck equation for the time evolution of the phase-space density.[1] Before using this equation to make predictions about the cooling, we will discuss the consequences of the necessary assumptions in order to facilitate a correct physical interpretation of the results. The Fokker-Planck description of stochastic cooling follows only in the next sections. In particular, the derivation of the Fokker-Planck equation is based on the following assumptions:

(1) The correlation time τ_{corr} of the signal fluctuations is finite. Contributions from the stochastic force $\delta F_j^R(\bar{\tau}_j)$ which are separated by time differences $\Delta t > \tau_{corr}$, then, do not have a definite phase relation with respect to each other and add up incoherently. After this time interval, they can be considered as statistically independent.

(2) The oscillation amplitudes of the particles change noticeably only after a time long compared with the correlation time of the fluctuations. Within a time interval τ_{corr} the amplitudes can be regarded almost as constant.

(3) Neglecting all collective particle effects, only the case $\langle F_j^R(\bar{\tau}_j)\rangle = 0$ is investigated.

According to Section 4.3, the requirement (1) is met by a finite width of the signal frequency distribution corresponding to a band spectrum of the fluctuations which is always present in an unbunched beam, owing to the energy spread of the particles. Likewise, stochastic cooling systems comply with the assumption (2) of *slow* cooling since typical cooling times span a range from a few seconds to many hours always larger than the reciprocal width of the relevant frequency bands [6]. On the other hand, the point (3) cannot be justified so easily; it can entail important consequences so that its meaning has to be considered in detail. We have already seen in Chapter 3 that a non-zero collective force $\langle F_j^R(\bar{\tau}_j)\rangle$ can lead to an unstable beam motion. In this situation, the collective force becomes the dominant term in the interaction, and obviously the assumption $\langle F_j^R(\bar{\tau}_j)\rangle = 0$ makes no sense. The verification of point (3) need to be based on the beam stability boundaries derived in Section 3.7: if the parameters of the cooling system only vary within these boundaries, all coherent modes of the beam will be damped, and thus will not contribute to the interaction.

Neglecting the collective force in Eq. (4.6), therefore, necessitates an *explicit* proof that the predictions for the cooling performance are *consistent* with the beam stability criteria of Section 3.7. A Fokker-Planck equation which relies on this assumption disregards any collective effect of the particles, and provides only purely statistical results. Hence such a description could predict that on average each beam particles is damped, although the collective beam motion is unstable. The Fokker-Planck equation describes the dynamics in a cooling system only partly and must be supplemented with the stability analysis elaborated in Chapter 3.

[1] At the beginning, the Fokker-Planck equation determines the probability density of each particle in the phase space. For statistically independent particles the probability densities of all particles are equal and can be identified with the mean phase-space density of the beam [14].

On the other hand, the damping of the coherent modes required by a stable beam motion introduces correlations among the particles which destroy the statistical independence of the particles. Now, the signal fluctuations arise around a macroscopic beam signal which is modulated by the collective interactions through the feedback loop so that the fluctuations become dynamically coupled with the collective beam motion. Since including these correlations into the mathematical description turns out to be very difficult, one simplifies the problem by assuming a small impact of this effect which hardly affects the time evolution of the phase-space density [8]. The particles are again considered uncorrelated. On the other hand, the damping of all coherent beam modes eliminates the collective force in the interaction, and enables us to set $\langle F_j^R(\bar{\tau}_j)\rangle = 0$ in Eq. (4.6). The interaction with the rest beam then reduces to the purely stochastic force $\delta F_j^R(\bar{\tau}_j)$.

Having manifested the underlying assumptions and limits of our cooling model, we now derive the time evolution of the phase-space density to predict the maximum attainable cooling rate. The parameters of the cooling system which give the best cooling results will be verified with respect to their compatibility with the beam stability in order to decide how far these values are physically reasonable.

4.5.2 The Fokker-Planck Equation

An appropriate set of variables for the description of transverse stochastic cooling is given by the action and angle variables (I, φ). Here, the action I is connected to the betatron amplitude A by the relation $I = A^2/2$ and φ is just the betatron phase. The time evolution of the corresponding distribution function $\bar{\rho}(I, \varphi, t)$ completely characterizes the cooling process, and obeys a two-dimensional Fokker-Planck equation. Since only the long-term behaviour of the action I is relevant for the cooling description, it is suitable to average the equation over the fast varying angle variable φ, resulting in the Fokker-Planck equation for the distribution function $\rho(I, t)$ [15]

$$\frac{\partial}{\partial t}\rho(I,t) = -\frac{\partial}{\partial I}\left\{F(I)\rho(I,t) - \frac{1}{2}D(I)\,\frac{\partial}{\partial I}\rho(I,t)\right\}. \tag{4.7}$$

The drift coefficient $F(I)$ and diffusion coefficient $D(I)$ are evaluated from the relations

$$F(I) = \Big\langle\Big\langle \frac{\Delta I^F}{\Delta T}\Big\rangle\Big\rangle_{\theta^0,\varphi^0} \qquad \text{and} \qquad D(I) = \Big\langle\Big\langle \frac{\Delta I^D \Delta I^D}{\Delta T}\Big\rangle\Big\rangle_{\theta^0,\varphi^0} \tag{4.8}$$

where ΔI^F denotes the change of the action during the time interval ΔT which arises from the coherent self-interaction of the particles, and ΔI^D is the corresponding value originating in the incoherent fluctuation signal of the other beam particles. The time interval ΔT must be chosen long compared with the correlation time of the fluctuations, but still much shorter than the time in which the density $\rho(I, t)$ changes appreciably due to the cooling. The brackets $\langle\langle\ \rangle\rangle$ in Eq. (4.8) indicate averaging processes over the initial azimuths θ^0 and betatron phases φ^0 of the beam particles. From the finite correlation time of the fluctuations follows that after a first delay $\Delta t > \tau_{corr}$ the initial state of the beam at the time $t = 0$ becomes insignificant. The starting-time and the initial conditions get an arbitrary meaning in so far as the system does not store any information about them, and the dynamics at times $t > \tau_{corr}$ no longer depends on the initial state of the beam. Hence not the initial conditions at the time $t = 0$ are important, but the corresponding values of the beam at the beginning of the averaging process. Because each averaging extents over a time interval $\Delta T > \tau_{corr}$,

the so defined *initial conditions* can be interpreted as stochastic quantities. In other words, the averaging process is the same after each time interval ΔT, and can always be performed over statistically independent phases and azimuths. By reason of the additional requirement $\Delta T < \tau_\rho$ the change of the phase-space density $\rho(I, t)$ during the averaging is negligible so that we consider the density $\rho(I, t)$ to remain constant over that period. Fig. 4-2 illustrates the relations between the different time intervals involved in the averaging processes.

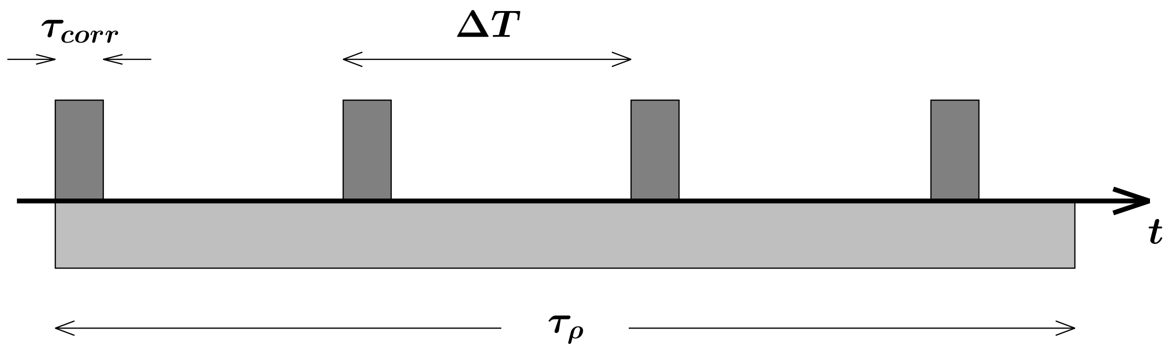

Figure 4-2: *Illustration of the different time scales of the averaging processes. The period ΔT over which the averaging is performed is large compared to the correlation time τ_{corr} of the fluctuations, but still short relative to the time τ_ρ in which the distribution changes noticeably.*

With the preceding remarks we can explicitly calculate the drift and diffusion coefficients of the test-particle j. The elementary, but lengthy derivations can be looked up in Appendix E, and yield the relations (4.9) and (4.10). The results are discussed in the Sections 4.5.3 and 4.5.4, revealing all the well-known properties of stochastic cooling in a quite natural way.

4.5.3 The Drift Coefficient

According to Eq. (E.6), we write the drift coefficient of the particle j as

$$F_j(I_j) = \bar{F}_j \, I_j \qquad \text{with} \qquad \bar{F}_j = -\frac{\kappa\omega_j}{Q} \sum_{(m)} |\widetilde{G}[(m+Q)\omega_j]| \, \sin\Phi_m^j \tag{4.9}$$

and the phase

$$\Phi_m^j = (m+Q)\theta^{PK} \frac{\delta\omega_j}{\omega_0} + \varphi^{PK}.$$

The drift coefficient describes the intrinsic damping of the action. It depends linearly on the gain $|\widetilde{G}(w)|$ of the cooling system, however weighted by a sine-function. The phase Φ_m^j already includes the time delay τ of the correction signals in the cooling electronics. Here, it has been assumed that this time delay is adjusted to the travel time of the synchronous particles (with the revolution frequency ω_0) from the pickup to the kicker, i.e. $\tau = \theta^{PK}/\omega_0$. These particles experience the maximum cooling effect if $\sin\Phi_m^j = 1$, which in consequence of $\delta\omega_j = 0$ is just the case for

$$\Phi_m^j = \frac{\pi}{2} + 2\pi n \quad \text{with} \quad n = 0, 1, 2, \ldots$$

This *optimum betatron phase advance* simply manifests the fact that the pickup measures the displacements of the particles, but the kicker corrects the offsets by applying the corresponding angles.

On the other hand, the particles with $\delta\omega_j \neq 0$ whose revolution frequencies vary from the nominal frequency ω_0 have different travel times from the pickup to the kicker, and so do not obtain their corrections with the optimum phase. The additional phase shift, often called *unwanted mixing*, is taken into account by the first term in the phase Φ^j_m. It follows from Eq. (4.6) that the unwanted mixing can be diminished for all particles simultaneously by reducing the azimuthal distance θ^{PK} between the pickup and kicker. For that reason stochastic cooling systems take a *shortcut through the storage ring*.

4.5.4 The Diffusion Coefficient

Neglecting the electronic noise in the cooling system, the diffusion coefficient of particle j reads (see Eq. (E.12))

$$D_j(I_j) = \bar{D}_j \langle I \rangle I_j$$

with

$$\bar{D}_j = \frac{2\pi\kappa^2\omega_j^2 N}{Q^2} \sum_{(m)} \left| \widetilde{G}[(m+Q)\omega_j] \right|^2 \sum_{(l)} \frac{1}{|l+Q|} f\left(\frac{m+Q}{l+Q}\,\omega_j \right) . \tag{4.10}$$

The diffusion coefficient is determined by the stochastic part of the interaction which originates in the signal fluctuations at the pickup. Due to the statistical nature of this interaction, the diffusion coefficient shows a quadratic dependence on the system gain, weighted by the spectral particle density at the frequencies at which the test-particle samples the correction signals. The diffusion increases the mean oscillation amplitudes of the particles, and so counteracts the cooling, hence it is also referred to as *heating* of the beam. In order to reduce the undesired diffusion without lowering the gain which at the same time would decrease the cooling, the cooling system has to operate in a frequency range where the frequency bands just begin to overlap, and the spectral particle density reaches its minimum value. We will discuss this requirement in detail. Defining in Eq. (4.10) the effective frequency distribution $\widehat{f}(\Omega)$ by

$$\widehat{f}(\Omega) = \sum_{(l)} \frac{1}{|l+Q|} f\left(\frac{\Omega}{l+Q} \right) , \tag{4.11}$$

the spectral particle density can be expressed as $dN/d\Omega = N\widehat{f}(\Omega)$. Owing to the internal summation over all frequency bands, particles from different bands can contribute to the value of $\widehat{f}$ at a given frequency Ω, and thus the definition (4.11) includes the case of overlapping frequency bands (see Sect. 2.6). For frequencies Ω at which the bands do not yet overlap only the band at the corresponding harmonic $m + Q \sim \Omega/\omega_0$ contributes in the summation in Eq. (4.11). For these harmonics the amplitude of $\widehat{f}(\Omega)$ decreases as $1/|m+Q|$ so that higher frequencies lead to the wanted reduction of the spectral particle density, as long as the frequency bands do not overlap. In the case of overlapping bands the contributions of the individual bands add up to a nearly constant value, and a further increase of the frequency would not leave any profit [5, 7].

We will visualize this behaviour in the time domain. For that purpose we consider the case in which neighboring bands just touch so that their width $\Delta\Omega_l$ is in the order of the revolution frequency: $\Delta\Omega_l \sim \omega_0$. Then the correlation time of the fluctuations is $\tau_{corr} \sim T_0$, i.e. signals which are sampled with the revolution time T_0 are statistically independent. The fluctuation

spectrum in such a frequency interval has the character of uncorrelated, white noise, similar to e.g. electronic noise. This situation corresponds to the so-called *good* or *perfect mixing*. Accordingly, *bad mixing* refers to the case of non-overlapping frequency bands in which the width of each band is smaller than the revolution frequency, and therefore the correlations last over more than one turn. Due to the larger amplitudes of the non-overlapping bands the bad mixing results in an enhanced diffusion. Improving the mixing, in general, necessitates broader frequency bands because then the particles within a single band spread over a larger frequency interval, and thus the spectral particle density becomes smaller.

The unwanted mixing in the drift coefficient (4.9), however, increases at higher harmonics. If the spectral particle density hardly changes in the frequency interval relevant to the stochastic cooling (perfect mixing), higher frequencies will only enhance the unwanted mixing, and hence will degrade the cooling. In case the undesired phase shifts of the particles exceed even the value $\pm\pi/2$, the particles are no longer damped but excited. Therefore one has to find an acceptable compromise between the two counteracting effects.

The results which we have derived so far are now used to make predictions for the cooling process.

4.5.5 The Cooling Rate

The cooling rate τ_{cool}^{-1} at the frequency ω_j follows from an integration of the Fokker-Planck equation (4.7) with the drift and diffusion coefficients given by Eqs. (4.9) and (4.10). The calculation in Appendix F.1 yields for the cooling rate the simple expression (F.1),

$$\tau_{cool}^{-1} = \bar{F} + \frac{1}{2}\bar{D}. \tag{4.12}$$

The relevance of the system bandwidth for the cooling rate becomes apparent from Eqs. (4.9) and (4.10). Both equations contain summations over all harmonics $(m+Q)\omega_j$, however only harmonics within the bandwidth of the cooling system contribute because only then $|\widetilde{G}[(m+Q)\omega_j]| \neq 0$. If for each of these harmonics the cooling effect exceeds the diffusion, there will remain net cooling contributions which all enter into the cooling rate (4.12). Because a larger bandwidth covers more harmonics, it results in a faster cooling.

The only parameter which permits to control the ratio of drift and diffusion coefficients is the gain $\widetilde{G}(w)$ of the cooling system (see Eqs. (4.9) and (4.10)). Of course we are interested in the value of the gain which results in the most efficient cooling operation, yielding the maximum cooling rate. According to Section 4.5.1, an arbitrary choice of the gain is only permitted within the limits of beam stability so that the predicted optimum value $\widetilde{G}_{opt}(w)$ has to be compared with the critical gain $\widetilde{G}_{crit}(w)$ in order to ensure that optimum cooling preserves the beam stability. We will satisfy this requirement in two steps:

- First, we calculate the optimum gain $\widetilde{G}_{opt}(w)$ for any possible frequency.
- For each frequency, we compare the optimum value with the critical gain $\widetilde{G}_{crit}(w)$ with regard to the beam stability.

The Optimum Gain

To calculate the optimum gain we consider the coefficients $\bar{F}$ and $\bar{D}$ in the cooling rate (4.12) at an arbitrary, but fixed frequency, and interpret them as pure functions of the gain. The

magnitude of the optimum gain $|\widetilde{G}_{opt}(\Omega_m)|$ at a given frequency $\Omega_m = (m+Q)\omega'$ is derived in Appendix F.2, and reads

$$\frac{1}{|\widetilde{G}_{opt}(\Omega_m)|} = \frac{2\kappa\omega' N}{Q}\,\pi \sum_{(l)} \frac{1}{|l+Q|}\, f\!\left(\frac{\Omega_m}{l+Q}\right). \tag{4.13}$$

Beam Stability under Optimum Cooling

Since the coherent beam modes completely describe the collective beam motion (see Sect. 3.7), they can be used to analyse the stability of the beam. To that end we interpret the frequency Ω_m as a mode frequency, and investigate how the critical and the optimum gain compare at this frequency. The magnitude of the critical gain $|\widetilde{G}_{crit}(\Omega_m)|$ is given by (3.31),

$$\frac{Q}{\kappa N|\widetilde{G}_{crit}(\Omega_m)|} = \pi \sum_{(l)} \frac{\cos\Phi_l^{\delta}(\Omega_m)}{|l+Q|}\frac{|\Omega_m|}{|l+Q|}\, f\!\left(\frac{\Omega_m}{l+Q}\right) + \sum_{(l)} \mathcal{P}\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\,\frac{\sin\Phi_l^{\mathcal{P}}(\Omega_m,\omega)}{(l+Q)\omega+\Omega_m}. \tag{4.14}$$

The phases $\Phi_l^{\delta}(\Omega_m)$ and $\Phi_l^{\mathcal{P}}(\Omega_m,\omega)$ in this expression contain the phase advance of the modes between the pickup and kicker so that the precise value of $|\widetilde{G}_{crit}(\Omega_m)|$ depends on the tune and the azimuthal distance from the pickup to the kicker. General results for the beam stability in a stochastic cooling system are difficult to obtain because the behaviour can largely vary depending on the actual choice of the parameters. At this place, we hence can only estimate the ratio of the critical and optimum gain by making a few reasonable assumptions. Supposing a symmetric frequency distribution $f(\omega)$ and a negligible shift of the real mode frequencies, $\delta\Omega_m \approx 0$, we expect only a small contribution from the principal value integral in Eq. (4.14) which can be neglected against the first term in this equation. Sizing the first term from above, we find a conservative criterion for the critical gain. Accordingly, we substitute for the cosine-function its maximum value: $\cos\Phi_l^{\delta}(\Omega_m) \to 1$, and obtain for Eq. (4.14)

$$\frac{Q}{\kappa N|\widetilde{G}_{crit}(\Omega_m)|} \leq \pi \sum_{(l)} \frac{1}{|l+Q|}\frac{|\Omega_m|}{|l+Q|}\, f\!\left(\frac{\Omega_m}{l+Q}\right). \tag{4.15}$$

The overlap condition for two adjacent frequency bands, say $m+Q$ and $l+Q = m+Q+\Delta l$ ($\Delta l = 0, \pm1, \pm2, \ldots$), is given by $(m+Q)\Delta\omega \sim \Delta l\,\omega_0$ where $\Delta\omega$ denotes the width of the frequency distribution around the center frequency ω_0. Hence the ratio of the two harmonic numbers can be written as

$$\frac{l+Q}{m+Q} = 1 + \frac{\Delta l}{m+Q} \sim 1 + \frac{\Delta\omega}{\omega_0} \sim 1 \pm 10^{-3}$$

which we approximate by one. With $|\Omega_m| = |m+Q|\omega'$ Eq. (4.15) then becomes

$$\frac{1}{|\widetilde{G}_{crit}(\Omega_m)|} \leq \frac{\pi\kappa\omega' N}{Q} \sum_{(l)} \frac{1}{|l+Q|}\, f\!\left(\frac{\Omega_m}{l+Q}\right).$$

A comparison with the optimum gain (4.13) immediately yields

$$\frac{1}{|\widetilde{G}_{crit}(\Omega_m)|} \leq \frac{1}{2|\widetilde{G}_{opt}(\Omega_m)|}.$$

Relying on the above approximations, we find that the optimum gain is at most half the critical gain so that beam stability is guaranteed over the entire frequency range. A similar result has been obtained earlier in [9] for the case of non-overlapping frequency bands. Of course one should realize that the result of this section gives only an estimate because it does not take into account the contribution of the principal value integral. Precise predictions will, in general, require numerical computations, using the actual parameters of the cooling system and storage ring.

Conclusion

The aim of this thesis was a description of stochastic cooling in the framework of control theory in order to allow predictions about the beam stability in such systems. This description has to treat the pickup and kicker of the cooling system as localized objects in the storage ring which impose a discrete time structure on both the generated signals and the forces experienced by the particles. The dynamics for a sampled interaction differ significantly from the motion under a force acting continuously in time, so that a careful stability analysis has to include the sampling.

The investigation presented here is based on the theory of multi-bunch feedback systems in which the control theory of discrete time signals had been adapted to feedback loops specific for accelerators. This special formulation takes strictly into account the positions of the pickup and kicker and the sampling of the interaction.

Relying on this formulation, a general stability criterion has been derived for an unbunched beam undergoing linear transverse stochastic cooling. The result allows for overlapping frequency bands in the beam spectrum and therefore goes beyond existing treatments. For the first time, the boundaries of beam stability could be predicted over the entire frequency range.

For the mathematical description of stochastic cooling a Fokker-Planck equation has been employed. This purely statistical approach does not include the collective motion of the particles and hence beam stability in the cooling system must be ensured. Owing to the results of this work, this prerequisite now can be verified. The stability criterion which has been obtained defines the boundaries within which the stability requirement is satisfied, and only there the Fokker-Planck equation produces physically reasonable results.

With a few assumptions a relation has been derived between the critical gain beyond which the beam motion becomes unstable and the optimum gain which allows the most efficient cooling operation. The comparison shows that a sufficient safety margin exists between these values, so it can be concluded that transverse stochastic cooling preserves beam stability even in the case of overlapping frequency bands. This result provides the Fokker-Planck approach to stochastic cooling with a well-founded physical and mathematical basis.

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Appendix A

Properties of the General Fourier Transformation

Here, we summarize only the properties of the general Fourier transformation used in the calculations of this work. Detailed information about the general Fourier transformation can be found e.g. in [19].

Let $f(t)$ and $g(t)$ be functions satisfying the condition (2.2). Their general Fourier transforms defined by (2.3) are denoted as $\widetilde{f}(w)$ and $\widetilde{g}(w)$ respectively. For the relation between the original function $f(t)$ and its transform $\widetilde{f}(w)$ we use the symbolic notation $\widetilde{f}(w) = \mathcal{F}[f(t)]$. Then the following properties can be derived [19]:

Damping Theorem Given a constant $\tau > 0$, then

$$\mathcal{F}[f(t-\tau)] = e^{-iw\tau}\widetilde{f}(w). \tag{A.1}$$

Displacement Theorem Let c be a complex constant. Then

$$\mathcal{F}\Big[e^{ict}f(t)\Big] = \widetilde{f}(w-c). \tag{A.2}$$

Differentiation Theorem Let $f^{(n)}(t)$ denote the n-th derivative of $f(t)$. Then

$$\mathcal{F}\Big[f^{(n)}(t)\Big] = (iw)^n\widetilde{f}(w) - \frac{(iw)^{n-1}}{2\pi}f(+0) - \frac{(iw)^{n-2}}{2\pi}f'(+0) - \ldots - \frac{1}{2\pi}f^{(n-1)}(+0) \tag{A.3}$$

where

$$f^{(n)}(+0) = \lim_{t\to+0} f^{(n)}(t).$$

Especially,

$$\mathcal{F}[f''(t)] = -w^2\widetilde{f}(w) - \frac{iw}{2\pi}f(+0) - \frac{1}{2\pi}f'(+0). \tag{A.4}$$

Convolution Theorem For the convolution $(f*g)$ of $f(t)$ and $g(t)$, defined by

$$(f*g)(t) = \int_0^t dt'\, f(t')g(t-t') \quad \text{with} \quad 0 \le t < \infty,$$

it follows that

$$\mathcal{F}[(f*g)(t)] = \widetilde{f}(w)\widetilde{g}(w). \tag{A.5}$$

Appendix B

The Harmonic Oscillator in the Formalism of the General Fourier Transformation

B.1 The Free Oscillator with Initial Conditions

Here, we consider a free harmonic oscillator with frequency Ω_0. To solve the equation of motion,

$$\ddot{x}(t) + \Omega_0^2\, x(t) = 0,$$

with initial conditions $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$ we transform this equation according to Eq. (2.3). Using the relation (A.4) this leads to

$$-\, w^2\, \widetilde{x}(w) + \Omega_0^2\, \widetilde{x}(w) - \frac{iw}{2\pi} x(0) - \frac{1}{2\pi}\dot{x}(0) = 0 \tag{B.1}$$

or

$$(-w^2 + \Omega_0^2)\, \widetilde{x}(w) = \frac{1}{2\pi}(iwx_0 + \dot{x}_0).$$

Hence follows

$$\widetilde{x}(w) = \frac{1}{2\pi}\frac{iwx_0 + \dot{x}_0}{(-w^2 + \Omega_0^2)} = \frac{1}{2\pi}\frac{iwx_0 + \dot{x}_0}{(-w + \Omega_0)(w + \Omega_0)}$$

so that the inverse transformation reads

$$x(t) = \int\limits_C dw\, \widetilde{x}(w) e^{iwt} = \frac{1}{2\pi}\int\limits_C dw\, \frac{iwx_0 + \dot{x}_0}{(-w + \Omega_0)(w + \Omega_0)}\, e^{iwt}.$$

The integrand has two poles on the real axis, $w_+ = +\Omega_0$ and $w_- = -\Omega_0$. To evaluate the integral, we close the integral contour in the upper w-plane, as is shown in Fig. B-1. The integral along the arc B gives no contribution because the integrand tends to zero for $|w| \longrightarrow \infty$. From Cauchy's residue theorem follows that only the residues of the poles w_+ and w_- contribute, yielding

$$x(t) = 2\pi i\, \mathrm{res}_{w_+}\{\widetilde{x}(w)\}\, e^{iw_+t} + 2\pi i\, \mathrm{res}_{w_-}\{\widetilde{x}(w)\}\, e^{iw_-t}. \tag{B.2}$$

For the residues we find

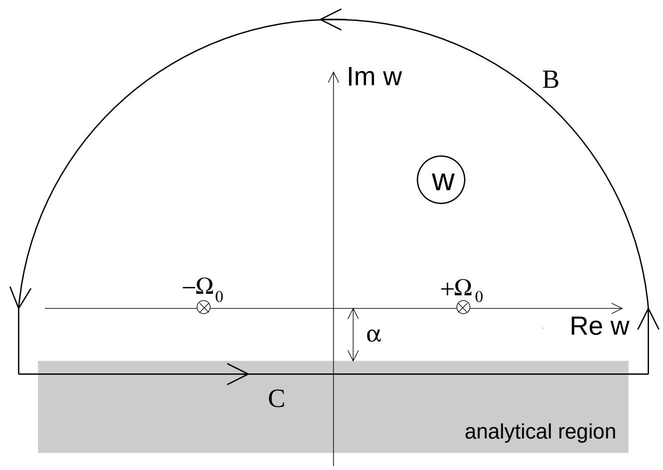

Figure B-1: *The poles of the free harmonic oscillator. To apply the residue theorem, we have closed the integral contour C in the upper w-plane.*

$$\mathrm{res}_{w_+}\{\tilde{x}(w)\}\,e^{iw_+t} = \lim_{w\to w_+}\{(w-w_+)\,\tilde{x}(w)\}\,e^{iw_+t} = -\frac{1}{2\pi}\,\frac{i\Omega_0 x_0 + \dot{x}_0}{2\Omega_0}\,e^{i\Omega_0 t}$$

and

$$\mathrm{res}_{w_-}\{\tilde{x}(w)\}\,e^{iw_-t} = \lim_{w\to w_-}\{(w-w_-)\,\tilde{x}(w)\}\,e^{iw_-t} = \frac{1}{2\pi}\,\frac{-i\Omega_0 x_0 + \dot{x}_0}{2\Omega_0}\,e^{-i\Omega_0 t}$$

so that Eq. (B.2) becomes

$$\begin{aligned} x(t) &= \frac{1}{2\Omega_0}\left\{(\Omega_0 x_0 - i\dot{x}_0)e^{i\Omega_0 t} + (\Omega_0 x_0 + i\dot{x}_0)e^{-i\Omega_0 t}\right\} \\ &= x_0\,\frac{1}{2}(e^{i\Omega_0 t} + e^{-i\Omega_0 t}) + \frac{\dot{x}_0}{\Omega_0}\,\frac{1}{2i}(e^{i\Omega_0 t} - e^{-i\Omega_0 t}). \end{aligned}$$

Finally, we get

$$x(t) = x_0\cos\Omega_0 t + \frac{\dot{x}_0}{\Omega_0}\sin\Omega_0 t.$$

Of course this is the expected result, but it has been derived without any intuitive ansatz. Furthermore the formalism includes the initial conditions from the beginning of the calculation (see Eq. (B.1)).

B.2 The Oscillator with Feedback Interaction

We will reconsider the self-interacting oscillator discussed in Section 2.4. According to Eq. (2.6), the equation of motion reads

$$\ddot{x}(t) + \Omega_0^2\,x(t) = \int_0^\infty dt'\,G(t-t')x(t') + g(t) \tag{B.3}$$

with the initial conditions $x(0) = \dot{x}(0) = 0$. Ω_0 denotes the frequency of the undisturbed oscillator. The impulse response $G(t)$ measures the strength of the feedback interaction, and $g(t)$ is the external δ-pulse excitation. Using the relations (A.4) and (A.5), the general Fourier transformation of Eq. (B.3) yields

$$(-w^2 + \Omega_0^2)\,\tilde{x}(w) = \tilde{G}(w)\tilde{x}(w) + \tilde{g}(w)$$

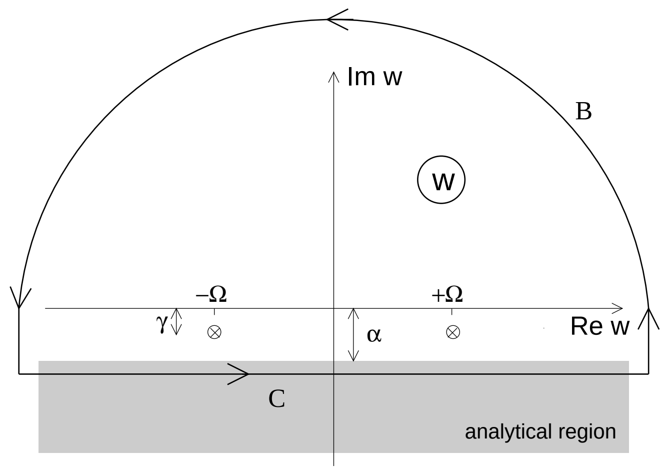

Figure B-2: *For $\gamma < 0$ the poles are located in the lower w-plane. The integral contour C has to run below the poles and is continued in the upper w-plane to a closed contour.*

where $\widetilde{g}(w) = A/\,2\pi$ is the transform of the δ-pulse $g(t) = A\delta(t)$. From this follows

$$\widetilde{x}(w) = \frac{1}{2\pi} \frac{A}{\left(-w^2 + \Omega_0^2 - \widetilde{G}(w)\right)}. \tag{B.4}$$

According to Eq. (2.4), the inverse transformation is given by

$$x(t) = \int_C dw\ \widetilde{x}(w) e^{iwt} = \frac{1}{2\pi} \int_C dw\ \frac{A}{-w^2 + \Omega_0^2 - \widetilde{G}(w)}\, e^{iwt}. \tag{B.5}$$

To proceed in this example, we assume an impedance $\widetilde{G}(w) = -i2\gamma w$ with $|\gamma| < \Omega_0$. Then Eq. (B.4) becomes

$$\widetilde{x}(w) = \frac{1}{2\pi} \frac{A}{(-w^2 + \Omega_0^2 + i2\gamma w)} = -\frac{1}{2\pi} \frac{A}{(w - w_+)(w - w_-)}$$

where

$$w_\pm = i\gamma \pm \Omega \qquad \text{with} \qquad \Omega = \sqrt{\Omega_0^2 - \gamma^2}.$$

With that, Eq. (B.5) can be written as

$$x(t) = -\frac{1}{2\pi} \int_C dw\ \frac{A}{(w - w_+)(w - w_-)}\, e^{iwt}. \tag{B.6}$$

The integrand has two poles, w_+ and w_-, in the upper or lower w-plane, depending on the sign of γ. The integral contour C can be chosen as a straight line with $\mathsf{Im}\, w < \gamma$, thus lying in the analytic region of $\widetilde{x}(w)$. This is shown in Fig. B-2 for the case $\gamma < 0$. The integral (B.6) is evaluated in the same way as in the previous section, i.e. the integral contour is closed in the upper w-plane by the arc B and the residue theorem is applied. This yields

$$x(t) = 2\pi i\, \mathrm{res}_{w_+}\{\widetilde{x}(w)\}\, e^{iw_+t} + 2\pi i\, \mathrm{res}_{w_-}\{\widetilde{x}(w)\}\, e^{iw_-t}. \tag{B.7}$$

With $w_+ - w_- = 2\Omega$, we obtain for the residues

$$\mathrm{res}_{w_+}\{\widetilde{x}(w)\}\, e^{iw_+t} = -\frac{1}{2\pi} \frac{A}{(w_+ - w_-)}\, e^{iw_+t} = -\frac{1}{2\pi} \frac{A}{2\Omega}\, e^{iw_+t}$$

and

$$\mathrm{res}_{w_-}\{\widetilde{x}(w)\}\, e^{iw_-t} = -\frac{1}{2\pi}\frac{A}{(w_- - w_+)}\, e^{iw_-t} = \frac{1}{2\pi}\frac{A}{2\Omega}\, e^{iw_-t}.$$

Since $iw_\pm = -\gamma \pm i\Omega$, Eq. (B.7) becomes

$$x(t) = -i\frac{A}{2\Omega}\left(e^{iw_+t} - e^{iw_-t}\right) = \frac{A}{\Omega}e^{-\gamma t}\,\frac{1}{2i}\left(e^{i\Omega t} - e^{-i\Omega t}\right)$$

and is finally written as

$$x(t) = \frac{A}{\Omega}e^{-\gamma t}\sin\Omega t \qquad \text{with} \qquad \Omega = \sqrt{\Omega_0^2 - \gamma^2}.$$

Depending on the sign of γ, we obtain an exponentially increasing ($\gamma < 0$) or decreasing ($\gamma > 0$) solution, i.e. the feedback interaction via an impedance can lead to an unstable motion of the oscillator.

Appendix C

The Derivation of the Self-consistent Solution for the Unbunched Beam

In this appendix, we will derive the self-consistent solution of the beam signal $\widetilde{S}(w)$ at the pickup, and the defining equation of the coefficients $\widetilde{C}_l(w)$. The calculation starts with the transformed equation of motion (3.23),

$$(-w^2 + \Omega_j^2)\, \widetilde{x}_j(w) = \Omega_j^2 \beta_K^{3/2} e^{-iw\tau_j^{PK}} \left(\widehat{F}_j(w) + \widehat{g}_j(w)\right).$$

Both sides of this equation are multiplied by $e^{-iwt_j^0}$ and divided by $(\Omega_j^2 - w^2)$, resulting in

$$\widetilde{x}_j(w) e^{-iwt_j^0} = \frac{e^{-iwt_j^0} e^{-iw\tau_j^{PK}}}{\Omega_j^2 - w^2} \Omega_j^2 \beta_K^{3/2} \left(\widehat{F}_j(w) + \widehat{g}_j(w)\right). \tag{C.1}$$

By inserting the decomposition (3.7) on the left-hand side and substituting $w \to w - m\omega_j$, this expression becomes

$$\frac{1}{N} \sum_l \widetilde{C}_l(w - m\omega_j) e^{il\theta_j^0} = \frac{e^{-i(w-m\omega_j)t_j^0} e^{-i(w-m\omega_j)\tau_j^{PK}}}{\Omega_j^2 - (w - m\omega_j)^2} \Omega_j^2 \beta_K^{3/2} \left(\widehat{F}_j(w - m\omega_j) + \widehat{g}_j(w - m\omega_j)\right).$$

Now, the periodicity of the functions $\widehat{F}_j(w)$ and $\widehat{g}_j(w)$ is used, and both sides of the equation are multiplied by $e^{-im\theta_j^0}$, followed by a summation over all particles j. Since $\theta_j^0 = \omega_j t_j^0$, this yields

$$\frac{1}{N} \sum_j \sum_l \widetilde{C}_l(w - m\omega_j) e^{i(l-m)\theta_j^0} = \sum_j \frac{e^{-iwt_j^0} e^{-i(w-m\omega_j)\tau_j^{PK}}}{\Omega_j^2 - (w - m\omega_j)^2} \Omega_j^2 \beta_K^{3/2} \left(\widehat{F}_j(w) + \widehat{g}_j(w)\right). \tag{C.2}$$

In this expression, we perform the transition to the continuous beam, as described in Section 3.4, i.e. the summation over the particles is replaced by an integration, following the instruction given by Eq. (3.8). On the left-hand side of Eq. (C.2), we get

$$\frac{1}{N} \sum_j \sum_l \widetilde{C}_l(w - m\omega_j) e^{i(l-m)\theta_j^0} \longrightarrow$$

$$\sum_{(l)} \int d\omega\ f_0(\omega) \widetilde{C}_l(w - m\omega) \int \frac{d\theta^0}{2\pi}\ e^{i(l-m)\theta^0} = \int d\omega\ f_0(\omega) \widetilde{C}_m(w - m\omega) \tag{C.3}$$

since

$$\int \frac{d\theta^0}{2\pi}\, e^{i(l-m)\theta^0} = \delta_{lm}.$$

The comments on page 26 permit us to write $\Omega_j = Q\omega_j$ and $\tau_j^{PK} = \varphi^{PK}/Q\omega_j$. Using Eqs. (3.19) and (3.20) and the definition $\bar{\kappa} = 2\pi RQ\sqrt{\beta_K}$, the first term $T_I(w)$ on the right-hand side of Eq. (C.2) containing the feedback force $\widehat{F}_j(w)$ reads

$$T_I(w) = \bar{\kappa} \sum_j \omega_j^2 \frac{e^{-i(w-m\omega_j)\varphi^{PK}/Q\omega_j}}{(Q\omega_j)^2 - (w-m\omega_j)^2} \sum_{(k)} \widetilde{G}(w+k\omega_j)\widetilde{S}(w+k\omega_j) e^{i(w+k\omega_j)\theta^{PK}/\omega_j} e^{ik\theta_j^0}.$$

After the transition to the continuous beam, the integration over the initial azimuths yields

$$\int \frac{d\theta^0}{2\pi}\, e^{ik\theta^0} = \delta_{k0}$$

so that only the term with $k = 0$ contributes to the summation. Hence follows

$$T_I(w) \longrightarrow \bar{\kappa} N \int d\omega\; f_0(\omega)\,\omega^2 \frac{e^{-i(w-m\omega)\varphi^{PK}/Q\omega}}{(Q\omega)^2 - (w-m\omega)^2} \widetilde{G}(w)\widetilde{S}(w) e^{iw\theta^{PK}/\omega}. \tag{C.4}$$

For the second term $T_{II}(w)$ on the right-hand side of Eq. (C.2) coming from the external excitation $\widehat{g}_j(w)$, the analogous calculation leads with Eqs. (3.21) and (3.22) to

$$T_{II}(w) \longrightarrow \bar{\kappa} N \int d\omega\; f_0(\omega)\,\omega^2 \frac{e^{-i(w-m\omega)\varphi^{PK}/Q\omega}}{(Q\omega)^2 - (w-m\omega)^2} \widetilde{g}(w) e^{iw\theta^{PK}/\omega}. \tag{C.5}$$

The combination of the results (C.3), (C.4) and (C.5) gives

$$\int d\omega\; f_0(\omega)\widetilde{C}_m(w-m\omega) = \bar{\kappa} N \int d\omega\; f_0(\omega)\,\omega^2 \frac{e^{-i(w-m\omega)\varphi^{PK}/Q\omega}}{(Q\omega)^2 - (w-m\omega)^2} e^{iw\theta^{PK}/\omega} \left(\widetilde{G}(w)\widetilde{S}(w) + \widetilde{g}(w)\right).$$

Multiplying by $\sqrt{\beta_P}/\,2\pi$, summing over all values m and using Eq. (3.9), we can write this equation as

$$\widetilde{S}(w) = \widetilde{R}(w)\widetilde{G}(w)\widetilde{S}(w) + \widetilde{R}(w)\widetilde{g}(w) \tag{C.6}$$

with the definitions $\kappa = \bar{\kappa}\cdot\sqrt{\beta_P}/\,2\pi$ and

$$\widetilde{R}(w) = \kappa N \int_0^\infty d\omega\; f_0(\omega)\,\omega^2 \sum_{(m)} \frac{e^{-i(w+m\omega)\varphi^{PK}/Q\omega}}{(Q\omega)^2 - (w+m\omega)^2}\, e^{iw\theta^{PK}/\omega}. \tag{C.7}$$

Solving Eq. (C.6) for $\widetilde{S}(w)$, we finally find the self-consistent solution to the coherent beam motion at the pickup:

$$\widetilde{S}(w) = \frac{\widetilde{R}(w)\widetilde{g}(w)}{1 - \widetilde{R}(w)\widetilde{G}(w)}. \tag{C.8}$$

From this result, the expansion coefficients $\widetilde{C}_l(w)$ in Eq. (3.7) can be derived. Starting at Eq. (C.1), we insert the decomposition (3.7) and obtain

$$\frac{1}{N}\sum_m \widetilde{C}_m(w) e^{im\theta_j^0} = \frac{e^{-iwt_j^0} e^{-iw\tau_j^{PK}}}{(Q\omega_j)^2 - w^2} (Q\omega_j)^2 \beta_K^{3/2} \left(\widehat{F}_j(w) + \widehat{g}_j(w)\right).$$

This equation is multiplied by $e^{-il\theta_j^0}$ and summed over all particles j, yielding

$$\frac{1}{N}\sum_j\sum_m \widetilde{C}_m(w)e^{i(m-l)\theta_j^0} = \sum_j \frac{e^{-iw\tau_j^{PK}}\,e^{-i(w+l\omega_j)t_j^0}}{(Q\omega_j)^2-w^2}\,(Q\omega_j)^2\beta_K^{3/2}\left(\widehat{F}_j(w)+\widehat{g}_j(w)\right). \tag{C.9}$$

Now we replace the summation over the particles by an integration, according to (3.8). On the left-hand side, the integral over θ^0 contributes only if $m = l$, and owing to the normalization $\int d\omega\, f_0(\omega) = 1$, we thus obtain

$$\frac{1}{N}\sum_j\sum_m \widetilde{C}_m(w)e^{i(m-l)\theta_j^0} \longrightarrow \widetilde{C}_l(w). \tag{C.10}$$

On the right-hand side of Eq. (C.9), the terms $T_I(w)$ and $T_{II}(w)$ containing the feedback force and the external excitation respectively are again treated separately. Using Eqs. (3.19) and (3.20), the feedback term reads

$$T_I(w) = \bar{\kappa}\sum_j \omega_j^2\,\frac{e^{-iw\varphi^{PK}/Q\omega_j}}{(Q\omega_j)^2-w^2}\sum_{(k)}\widetilde{G}(w+k\omega_j)\widetilde{S}(w+k\omega_j)e^{i(w+k\omega_j)\theta^{PK}/\omega_j}e^{i(k-l)\theta_j^0}$$

so that the transition to the continuous beam leads to

$$T_I(w) \longrightarrow \bar{\kappa}N\int d\omega\; f_0(\omega)\,\omega^2\,\frac{e^{-iw\varphi^{PK}/Q\omega}}{(Q\omega)^2-w^2}\,\widetilde{G}(w+l\omega)\widetilde{S}(w+l\omega)e^{i(w+l\omega)\theta^{PK}/\omega}. \tag{C.11}$$

Similarly, we obtain for the external excitation

$$T_{II}(w) \longrightarrow \bar{\kappa}N\int d\omega\; f_0(\omega)\,\omega^2\,\frac{e^{-iw\varphi^{PK}/Q\omega}}{(Q\omega)^2-w^2}\,\widetilde{g}(w+l\omega)e^{i(w+l\omega)\theta^{PK}/\omega}. \tag{C.12}$$

Combining the Eqs. (C.10), (C.11) and (C.12) results in

$$\widetilde{C}_l(w) = \bar{\kappa}N\int d\omega\; f_0(\omega)\,\omega^2\,\frac{e^{-iw\varphi^{PK}/Q\omega}}{(Q\omega)^2-w^2}\,e^{i(w+l\omega)\theta^{PK}/\omega}\left(\widetilde{G}(w+l\omega)\widetilde{S}(w+l\omega)+\widetilde{g}(w+l\omega)\right).$$

Inserting here the self-consistent solution (C.8) reduces the expression in the brackets. Writing $w' = w + l\omega$, it can easily be shown that

$$\widetilde{G}(w')\widetilde{S}(w')+\widetilde{g}(w') = \widetilde{G}(w')\,\frac{\widetilde{R}(w')\widetilde{g}(w')}{1-\widetilde{G}(w')\widetilde{R}(w')}+\widetilde{g}(w') = \frac{\widetilde{g}(w')}{1-\widetilde{G}(w')\widetilde{R}(w')}.$$

From this finally follows

$$\widetilde{C}_l(w) = \bar{\kappa}N\int_0^\infty d\omega\; f_0(\omega)\,\omega^2\,\frac{e^{-iw\varphi^{PK}/Q\omega}}{(Q\omega)^2-w^2}\,\frac{\widetilde{g}(w+l\omega)e^{i(w+l\omega)\theta^{PK}/\omega}}{1-\widetilde{G}(w+l\omega)\widetilde{R}(w+l\omega)}. \tag{C.13}$$

Appendix D

The Derivation of the Beam Stability Criteria

Starting at the stability condition (3.29), we will present the calculation which lead to the stability criteria (3.31) and (3.32). For the evaluation of the limit in Eq. (3.29), it is convenient to define a new distribution function $f(\omega)$ by

$$f(\omega) = \frac{1}{2}\Big\{f_0(\omega) + f_0(-\omega)\Big\} \qquad \text{with} \qquad \int\limits_{-\infty}^{+\infty} d\omega\; f(\omega) = 1 \quad \text{and} \quad f(-\omega) = f(\omega). \tag{D.1}$$

Using this definition, we can write the Eq. (3.25) in the following form:

$$\begin{aligned} \widetilde{R}(w) &= \frac{\kappa N}{2Q} \sum_{(\pm)} \sum_{(m)} \int\limits_{0}^{\infty} d\omega\; f_0(\omega)\,\omega\, \frac{\pm e^{-iw\Delta\varphi/Q\omega}}{w + (m \pm Q)\omega}\, e^{-im\varphi^{PK}/Q} \\ &= \frac{\kappa N}{Q} \sum_{(m)} \int\limits_{-\infty}^{\infty} d\omega\; f(\omega)\,\omega\, \frac{e^{-i\sigma_\omega w\Delta\varphi/Q\omega}}{w + (m + Q)\omega}\, e^{-i\sigma_\omega m\varphi^{PK}/Q} \end{aligned}$$

where $\sigma_\omega = \omega/|\omega|$ indicates the sign of ω, and $\Delta\varphi = \varphi^{PK} - Q\theta^{PK}$. Then $\widetilde{R}(\Omega_l + i\gamma_l)$ can be expressed as

$$\begin{aligned} \widetilde{R}(\Omega_l + i\gamma_l) &= \frac{\kappa N}{Q} \sum_{(m)} \int\limits_{-\infty}^{\infty} d\omega\; f(\omega)\,\omega\, \frac{e^{-i\sigma_\omega \Omega_l \Delta\varphi/Q\omega}\; e^{\gamma_l \sigma_\omega \Delta\varphi/Q\omega}}{\Omega_l + (m+Q)\omega + i\gamma_l}\, e^{-i\sigma_\omega m\varphi^{PK}/Q} \\ &= \frac{\kappa N}{Q} \sum_{(m)} \frac{1}{m+Q} \int\limits_{-\infty}^{\infty} d\omega\; f(\omega)\,\omega\, \frac{e^{-i\sigma_\omega \Omega_l \Delta\varphi/Q\omega}\; e^{\gamma_l \sigma_\omega \Delta\varphi/Q\omega}}{\omega + \frac{\Omega_l}{m+Q} + i\sigma_{\underline{m}} \frac{\gamma_l}{|m+Q|}}\, e^{-i\sigma_\omega m\varphi^{PK}/Q}. \end{aligned}$$

Here, $\sigma_{\underline{m}} = (m+Q)/|m+Q|$ gives the sign of $(m+Q)$. The limit $\gamma_l \to 0^+$ of $\widetilde{R}(\Omega_l + i\gamma_l)$ can be computed using the identities

$$\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = \mathcal{P}(\frac{1}{x}) \mp i\pi\delta(x)$$

where $\mathcal{P}(\frac{1}{x})$ denotes the principal value. In the present case, we obtain

$$\lim_{\gamma_l \to 0^+} \widetilde{R}(\Omega_l + i\gamma_l) = \frac{\kappa N}{Q}\left\{I_{\mathcal{P}}(\Omega_l) - i\pi I_\delta(\Omega_l)\right\} \tag{D.2}$$

with the principal value integral

$$I_{\mathcal{P}}(\Omega_l) = \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\ f(\omega)\,\omega\,\frac{e^{-i\sigma_\omega \Omega_l \Delta\varphi/Q\omega}}{(m+Q)\omega+\Omega_l}\ e^{-i\sigma_\omega m\varphi^{PK}/Q}, \tag{D.3}$$

and the integral containing the δ-function

$$I_\delta(\Omega_l) = \sum_{(m)} \frac{\sigma_{\underline{m}}}{m+Q} \int_{-\infty}^{\infty} d\omega\ f(\omega)\,\omega\, e^{-i\sigma_\omega \Omega_l \Delta\varphi/Q\omega}\ e^{-i\sigma_\omega m\varphi^{PK}/Q}\ \delta\!\left(\omega + \frac{\Omega_l}{m+Q}\right).$$

Contributions from the δ-function only arise at the frequencies $\omega = -\Omega_l/(m+Q)$ so that $\sigma_\omega = -\sigma_{\Omega_l}\sigma_{\underline{m}}$, depending on the sign σ_{Ω_l} of Ω_l. Together with $\sigma_{\underline{m}}(m+Q) = |m+Q|$ and $f(-\omega) = f(\omega)$, this leads to

$$I_\delta(\Omega_l) = -\sum_{(m)} \frac{1}{|m+Q|}\ f\!\left(\frac{\Omega_l}{m+Q}\right)\ \frac{\Omega_l}{m+Q}\ e^{i\sigma_{\Omega_l}\sigma_{\underline{m}}(m\theta^{PK}-\Delta\varphi)}. \tag{D.4}$$

Combining the Eqs. (3.29), (D.2), (D.3) and (D.4), we obtain the defining equation for the critical gain $\widetilde{G}_{crit}(w)$:

$$\begin{aligned} 1 \quad = \quad & \widetilde{G}_{crit}(\Omega_l)\,\frac{i\pi\kappa N}{Q} \sum_{(m)} \frac{1}{|m+Q|}\ f\!\left(\frac{\Omega_l}{m+Q}\right)\ \frac{\Omega_l}{m+Q}\ e^{i\sigma_{\Omega_l}\sigma_{\underline{m}}(m\theta^{PK}-\Delta\varphi)} + \\ & \widetilde{G}_{crit}(\Omega_l)\,\frac{\kappa N}{Q} \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\ f(\omega)\,\omega\,\frac{e^{-i\sigma_\omega \Omega_l \Delta\varphi/Q\omega}}{(m+Q)\omega+\Omega_l}\ e^{-i\sigma_\omega m\varphi^{PK}/Q}. \end{aligned}$$

Expressing the critical gain in terms of amplitude and phase, i.e. $\widetilde{G}_{crit}(w) = |\widetilde{G}_{crit}(w)|e^{i\psi(w)}$, the above equation can be written as

$$\begin{aligned} \frac{Q}{\kappa N|\widetilde{G}_{crit}(\Omega_l)|} \quad = \quad & i\pi \sum_{(m)} \frac{1}{|m+Q|}\ f\!\left(\frac{\Omega_l}{m+Q}\right)\ \frac{\Omega_l}{m+Q}\ e^{i\bar{\Phi}^{\delta}_m(\Omega_l)} + \\ & \sum_{(m)} \mathcal{P}\!\!\int_{-\infty}^{\infty} d\omega\ f(\omega)\,\omega\,\frac{e^{i\bar{\Phi}^{\mathcal{P}}_m(\Omega_l,\omega)}}{(m+Q)\omega+\Omega_l} \end{aligned} \tag{D.5}$$

where the phases are

$$\bar{\Phi}^{\delta}_m(\Omega_l) = \psi(\Omega_l) + \sigma_{\Omega_l}\sigma_{\underline{m}}\,(m\theta^{PK} - \Delta\varphi) \tag{D.6}$$

and

$$\bar{\Phi}^{\mathcal{P}}_m(\Omega_l,\omega) = \psi(\Omega_l) - \sigma_\omega\,\Omega_l\Delta\varphi/Q\omega - \sigma_\omega\, m\varphi^{PK}/Q. \tag{D.7}$$

The assumptions about stochastic cooling systems (see page 28) allow the further evaluation of the phases $\bar{\Phi}^{\delta}_m(\Omega_l)$ and $\bar{\Phi}^{\mathcal{P}}_m(\Omega_l,\omega)$. With $\omega' = \omega_0 + \delta\omega'$ and $\Delta\varphi = \varphi^{PK} - Q\theta^{PK}$, it follows for Eq. (D.6) that

$$\begin{aligned} \bar{\Phi}^{\delta}_m(\Omega_l) \quad & \approx \quad -(l+Q)\theta^{PK}\frac{\omega'}{\omega_0} + \sigma_{\underline{l}}\sigma_{\underline{m}}\,(m\theta^{PK} - \varphi^{PK} + Q\theta^{PK}) \\ & \approx \quad -\sigma_{\underline{l}}\sigma_{\underline{m}}\,\varphi^{PK} - \Big[(l+Q) - \sigma_{\underline{l}}\sigma_{\underline{m}}\,(m+Q)\Big]\theta^{PK} - (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0} \\ & \approx \quad -\sigma_{\underline{l}}\sigma_{\underline{m}}\,\varphi^{PK} - \sigma_{\underline{l}}\,\Big[|l+Q| - |m+Q|\Big]\theta^{PK} - (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}. \end{aligned}$$

Defining the relative frequency difference $\Delta'(\omega) = (\omega' - |\omega|)/|\omega|$ of a frequency ω from the frequency ω', Eq. (D.7) becomes

$$
\begin{aligned}
\bar{\Phi}_m^{\mathcal{P}}(\Omega_l, \omega) &\approx -(l+Q)\theta^{PK}\frac{\omega'}{\omega_0} - \sigma_\omega\,(l+Q)\frac{\Delta\varphi}{Q}\frac{\omega'}{\omega} - \sigma_\omega\, m\frac{\varphi^{PK}}{Q} \\
&\approx -(l+Q)\theta^{PK}(1+\frac{\delta\omega'}{\omega_0}) - (l+Q)\frac{\Delta\varphi}{Q}(1+\Delta'(\omega)) - \sigma_\omega\,(m+Q)\frac{\varphi^{PK}}{Q} + \sigma_\omega\,\varphi^{PK} \\
&\approx \sigma_\omega\,\varphi^{PK} - \Big[(l+Q) + \sigma_\omega\,(m+Q)\Big]\frac{\varphi^{PK}}{Q} - (l+Q)\frac{\Delta\varphi}{Q}\Delta'(\omega) - (l+Q)\theta^{PK}\frac{\delta\omega'}{\omega_0}.
\end{aligned}
$$

It is convenient to write the phases as $\bar{\Phi}_m^{\delta} = -\sigma_{\underline{l}}\sigma_{\underline{m}}\varphi^{PK} - \Phi_m^{\delta}$ and $\bar{\Phi}_m^{\mathcal{P}} = \sigma_\omega\varphi^{PK} - \Phi_m^{\mathcal{P}}$ because then the betatron phase advance φ^{PK} can be considered explicitly. According to the remarks on page 28, we set $\varphi^{PK} = \pi/2 + 2\pi n, n = 0, 1, \ldots$, and obtain for the real and imaginary parts of Eq. (D.5)

$$
\frac{Q}{\kappa N|\widetilde{G}_{crit}(\Omega_l)|} = \pi\sum_{(m)}\frac{\cos\Phi_m^{\delta}(\Omega_l)}{|m+Q|}\frac{|\Omega_l|}{|m+Q|}\,f\!\left(\frac{\Omega_l}{m+Q}\right) + \sum_{(m)}\mathcal{P}\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\,\frac{\sin\Phi_m^{\mathcal{P}}(\Omega_l,\omega)}{(m+Q)\omega+\Omega_l} \tag{D.8}
$$

and

$$
\pi\sum_{(m)}\frac{\sin\Phi_m^{\delta}(\Omega_l)}{|m+Q|}\frac{|\Omega_l|}{|m+Q|}\,f\!\left(\frac{\Omega_l}{m+Q}\right) = \sum_{(m)}\mathcal{P}\!\int_{-\infty}^{\infty} d\omega\; f(\omega)\,|\omega|\,\frac{\cos\Phi_m^{\mathcal{P}}(\Omega_l,\omega)}{(m+Q)\omega+\Omega_l}. \tag{D.9}
$$

These equations are the basic relations for the stability analysis carried out in this work.

Appendix E

The Calculation of the Drift and Diffusion Coefficients

In this section, we will derive the drift and diffusion coefficients of the Fokker-Planck equation (4.7). To the order of approximation, the evaluation of the expressions in (4.8) uses the unperturbed zero-order particle trajectories. In this case the transverse motion can be written as[1]

$$x(\bar{\tau}) = A\cos(\Omega\bar{\tau} + \varphi^0) \qquad \text{and} \qquad \dot{x}(\bar{\tau}) = -A\Omega\,\sin(\Omega\bar{\tau} + \varphi^0)$$

where A, $\Omega = Q\omega$ and φ^0 are the amplitude, frequency and initial phase of the betatron oscillation respectively. The action $I(\bar{\tau})$ of a particle along its unperturbed trajectory is given by

$$I(\bar{\tau}) = \frac{1}{2}\left\{x^2(\bar{\tau}) + \frac{1}{\Omega^2}\,\dot{x}^2(\bar{\tau})\right\}$$

from which immediately follows that

$$\dot{I}(\bar{\tau}) = \frac{1}{\Omega^2}\,\dot{x}(\bar{\tau})\left\{\ddot{x}(\bar{\tau}) + \Omega^2 x(\bar{\tau})\right\} = \frac{1}{\Omega^2}\,\dot{x}(\bar{\tau})K(\bar{\tau})$$

where $K(\bar{\tau})$ denotes the force acting on the particle. The change ΔI of the action after a time ΔT can be obtained by integrating $\dot{I}(\bar{\tau})$ over this time interval which then allows a calculation of the drift and diffusion coefficients. Again it should be pointed out that this approximation implicitly presumes a stable beam motion, and therefore the expressions for the drift and diffusion coefficients are only valid within the boundaries of beam stability. The consequences arising from this are discussed in the introduction and in Section 4.5.1.

The Drift Coefficient

The drift coefficient of the particle j is given by

$$F_j(I_j) = \left\langle\left\langle \frac{\Delta I_j^F}{\Delta T} \right\rangle\right\rangle_{\varphi_j^0}. \tag{E.1}$$

The change ΔI_j^F of the action experienced by the particle in a time interval ΔT due to its self-interaction can be evaluated from

$$\Delta I_j^F = \frac{1}{\Omega^2}\int_0^{\Delta T} d\bar{\tau}_j\;\dot{x}(\bar{\tau}_j)F_j^S(\bar{\tau}_j). \tag{E.2}$$

[1] A dot on top of a variable symbolizes the derivative with respect to the quasi-time $\bar{\tau}$.

According to Eq. (4.4), the self-force $F_j^S(\bar\tau_j)$ reads

$$F_j^S(\bar\tau_j) = \kappa\omega_j^2 \int dw\; e^{iw\bar\tau_j} e^{-iw\tau_j^{PK}} \widehat{G}_j(w) \sum_{(l)} \widetilde{x}_j(w + l\omega_j) \tag{E.3}$$

with the periodic function $\widehat{G}_j(w)$ defined by

$$\widehat{G}_j(w) = \sum_{(m)} \widetilde{G}(w + m\omega_j)\, e^{i(w+m\omega_j)t_j^{PK}}.$$

The transform of the unperturbed motion has already been derived in Section B.1, yielding

$$\widetilde{x}_j(w) = \frac{1}{2\pi}\frac{iwx_j^0 + \dot{x}_j^0}{\Omega_j^2 - w^2} \qquad \text{with} \qquad x_j^0 = A_j \cos\varphi_j^0 \quad \text{and} \quad \dot{x}_j^0 = -A_j\Omega_j \sin\varphi_j^0. \tag{E.4}$$

The integral over w in Eq. (E.3) only contributes at the poles of the function $\widetilde{x}_j(w)$ (see Sect. B.1), and the elementary calculation leads to

$$F_j^S(\bar\tau_j) = \frac{\kappa\omega_j^2}{2} A_j \sum_{\pm}\sum_{(l)} \widehat{G}_j(\pm\Omega_j)\, e^{-i(l\omega_j \pm\Omega_j)\tau_j^{PK}} e^{\pm i\varphi_j^0} e^{i(l\omega_j\pm\Omega_j)\bar\tau_j}.$$

Inserting this expression into Eq. (E.2) and recalling $I_j = A_j^2/2$, we obtain

$$\Delta I_j^F = \frac{\kappa\omega_j}{2iQ} I_j \sum_{\pm'}\sum_{\pm}\sum_{(l)} (\mp')\, \widehat{G}_j(\pm\Omega_j)\, e^{-i(l\omega_j\pm\Omega_j)\tau_j^{PK}} e^{\pm i\varphi_j^0} e^{\pm' i\varphi_j^0} \int_0^{\Delta T} d\bar\tau_j\; e^{i(l\omega_j\pm\Omega_j\pm'\Omega_j)\bar\tau_j}.$$

The time integration can easily be carried out, resulting in

$$\int_0^{\Delta T} d\bar\tau_j\; e^{i(l\omega_j\pm\Omega_j\pm'\Omega_j)\bar\tau_j} = \frac{\sin(l\omega_j \pm \Omega_j \pm' \Omega_j)\Delta T/2}{(l\omega_j \pm \Omega_j \pm' \Omega_j)/2}\; e^{i(l\omega_j\pm\Omega_j\pm'\Omega_j)\Delta T/2}.$$

Since we assumed that $\Delta T \gg T_0$, a significant contribution from this expression arises only if $l\omega_j \pm \Omega_j \pm' \Omega_j \approx 0$, thus requiring $l = 0$ *and* $\pm\Omega_j = \mp'\Omega_j$ at the same time. Hence follows

$$\frac{\sin(l\omega_j \pm \Omega_j \pm' \Omega_j)\Delta T/2}{(l\omega_j \pm \Omega_j \pm' \Omega_j)/2}\; e^{i(l\omega_j\pm\Omega_j\pm'\Omega_j)\Delta T/2} \longrightarrow \Delta T \qquad \text{for} \qquad l\omega_j \pm \Omega_j \pm' \Omega_j \longrightarrow 0.$$

Since $\varphi^{PK} = \Omega_j\tau_j^{PK}$ and $\theta^{PK} = \omega_j t_j^{PK}$, the drift coefficient (E.1) then reads

$$F_j(I_j) = \bar{F}_j\, I_j$$

with

$$\bar{F}_j = \frac{\kappa\omega_j}{2iQ} \sum_{\pm}\sum_{(m)} (\pm)\, \widetilde{G}(m\omega_j \pm \Omega_j)\, e^{i(m\pm Q)\theta^{PK}} e^{\mp i\varphi^{PK}}. \tag{E.5}$$

According to the comments about stochastic cooling sytems on page 28, we assume

$$\widetilde{G}(\omega) = |\widetilde{G}(\omega)|\, e^{-i\omega\tau} \qquad \text{with} \qquad \tau = \theta^{PK}/\omega_0.$$

Writing the frequency difference of the particle j from the nominal frequency ω_0 by $\delta\omega_j = \omega_j - \omega_0$ and using the relation $\widetilde{G}(-\omega) = \widetilde{G}^*(\omega)$ (see Sect. 2.2), we obtain for Eq. (E.5)

$$\bar{F}_j = -\frac{\kappa\omega_j}{Q} \sum_{(m)} \left|\widetilde{G}[(m+Q)\omega_j]\right| \sin\Phi_m^j \tag{E.6}$$

with the phase

$$\Phi_m^j = (m+Q)\theta^{PK}\frac{\delta\omega_j}{\omega_0} + \varphi^{PK}.$$

The Diffusion Coefficient

The diffusion coefficient of the particle j in Eq. (4.8) is derived from

$$D_j(I_j) = \Big\langle\Big\langle \frac{\Delta I_j^D \Delta I_j^D}{\Delta T} \Big\rangle\Big\rangle_{\varphi_j^0}. \tag{E.7}$$

Here, ΔI_j^D describes the change of the action resulting from the interaction with the rest beam during the time ΔT, given by

$$\Delta I_j^D = \frac{1}{\Omega^2} \int_0^{\Delta T} d\bar{\tau}_j \; \dot{x}(\bar{\tau}_j) F_j^R(\bar{\tau}_j). \tag{E.8}$$

According to Eq. (4.5), the expression for $F_j^R(\bar{\tau}_j)$ reads

$$F_j^R(\bar{\tau}_j) = \kappa\omega_j^2 \int dw \; e^{iw\bar{\tau}_j} e^{-iw\tau_j^{PK}} \sum_{(m)} \widetilde{G}(w + m\omega_j)\, \widetilde{S}_j(w + m\omega_j) e^{i(w+m\omega_j)t_j^{PK}} e^{i(w+m\omega_j)t_j^0}$$

with

$$\widetilde{S}_j(w) = \sum_{j' \neq j} \sum_{(l)} \widetilde{x}_{j'}(w + l\omega_{j'}) e^{-iwt_{j'}^0}.$$

Again $\widetilde{x}_{j'}(w)$ denotes the transform of the unperturbed motion given by (E.4). As in the case of the drift coefficient, only the poles of $\widetilde{x}_{j'}(w)$ contribute in the w-integration of the previous equation. Performing this integration and inserting the result into Eq. (E.8) followed by the remaining integration over the time $\bar{\tau}_j$, we obtain the expression

$$\begin{aligned} \Delta I_j^D \;=\; & \frac{\kappa\omega_j}{2iQ} A_j \sum_{j' \neq j} \sum_{(m)} \sum_{(l)} \sum_{\pm} \sum_{\pm'} (\pm)\, e^{\pm i\varphi_j^0} \widetilde{G}(-l\omega_{j'} \pm' \Omega_{j'}) e^{-i(l\omega_{j'} \mp' \Omega_{j'})t_j^{PK}} \\ & \frac{\sin(m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'})\Delta T/2}{(m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'})/2} \, e^{i(m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'})\Delta T/2} \\ & e^{-i(m\omega_j - l\omega_{j'} \pm' \Omega_{j'})\tau_j^{PK}} e^{-i(l\omega_{j'} \mp' \Omega_{j'})t_j^0} e^{i(l\omega_{j'} \mp' \Omega_{j'})t_{j'}^0} \frac{A_{j'}}{2} e^{\pm' i\varphi_{j'}^0}. \end{aligned} \tag{E.9}$$

For the calculation of the diffusion coefficient (E.7), two expressions (E.9) are multiplied having independent summation indices, say $\{j', m, l, \pm, \pm'\}$ and $\{j'', m', l', \mp, \pm''\}$. This product is averaged over the betatron phases and azimuths of the particles at the beginning of the time integration. According to the arguments in Section 4.5.2, these variables can be considered as statistically independent. Averaging over the betatron phase φ_j^0 of the test-particle, we encounter expressions of the form

$$\Big\langle e^{\pm i\varphi_j^0} e^{\mp i\varphi_j^0} \Big\rangle_{\varphi_j^0} = \delta_{\pm\mp}$$

giving contributions only if the phases cancel each other. The second averaging process involves both the betatron phases and azimuths of the other beam particles. For uncorrelated, equally distributed betatron phases, it follows that

$$\Big\langle e^{\pm' i\varphi_{j'}^0} e^{\pm'' i\varphi_{j''}^0} \Big\rangle_{\varphi_{j'}^0, \varphi_{j''}^0} = \delta_{\pm'\mp''}\, \delta_{j'j''}.$$

With that, the remaining averaging over the azimuths yields

$$\left\langle e^{il\theta^0_{j'}} e^{il'\theta^0_{j'}} \right\rangle_{\theta^0_{j'}} = \delta_{l,-l'}$$

so that after all we obtain

$$\begin{aligned} D_j(I_j) \;&=\; \frac{1}{\Delta T}\frac{\kappa^2\omega_j^2}{4Q^2}\, I_j \sum_{j'\neq j}\sum_{(m)}\sum_{(m')}\sum_{(l)}\sum_{\pm}\sum_{\pm'} \left|\widetilde{G}(l\omega_{j'} \mp' \Omega_{j'})\right|^2 I_{j'} \\ &\quad \frac{\sin(m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'})\Delta T/2}{(m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'})/2}\, e^{im\omega_j \Delta T/2} e^{-im\omega_j \tau_j^{PK}} \\ &\quad \frac{\sin(m'\omega_j \mp \Omega_j + l\omega_{j'} \mp' \Omega_{j'})\Delta T/2}{(m'\omega_j \mp \Omega_j + l\omega_{j'} \mp' \Omega_{j'})/2}\, e^{im'\omega_j \Delta T/2} e^{-im'\omega_j \tau_j^{PK}} . \end{aligned} \tag{E.10}$$

The value of $D_j(I_j)$ will differ significantly from zero only if the frequency differences in this expression obey the conditions

and

$$\begin{aligned} -1/\Delta T &< m\omega_j \pm \Omega_j - l\omega_{j'} \pm' \Omega_{j'} < 1/\Delta T \\ -1/\Delta T &< m'\omega_j \mp \Omega_j + l\omega_{j'} \mp' \Omega_{j'} < 1/\Delta T. \end{aligned}$$

Adding both equations yields

$$-2/\Delta T < (m + m')\omega_j < 2/\Delta T.$$

According to Section 4.5.2, the time interval ΔT is much larger than the correlation time τ_{corr}, and thus $\Delta T \gg T_0$ so that both requirements can be satisfied together only by $m' = -m$.

We now replace in Eq. (E.10) the summation over the particles j' by an integration over the corresponding distribution function $\bar{f}(\omega, I)$. Provided the frequency ω and the action I are independent variables, this distribution function factorizes and can be written as $\bar{f}(\omega, I) = f(\omega)\rho(I)$. The normalized distribution functions obey the conditions[2]

$$\int_0^\infty dI\ \rho(I) = 1, \qquad \int_{-\infty}^{\infty} d\omega\ f(\omega) = 1 \quad \text{and} \quad f(-\omega) = f(\omega).$$

Then the integration over the action can be carried out immediately, yielding

$$\int_0^\infty dI\ I\rho(I) = \langle I \rangle$$

where it has been presumed that the distribution $\rho(I)$ does not change over the time interval ΔT. This assumption is justified since ΔT is small compared to the time in which $\rho(I)$ alters noticeably due to the cooling, i.e. $\Delta T \ll \tau_{cool}$. Performing the integration over the action, we are hence permitted to use a time-independent, constant distribution $\rho(I)$, determined by its value at the beginning of the time interval ΔT.

Writing the betatron frequency as $\Omega = Q\omega$ and using the relations $\widetilde{G}^*(\omega) = \widetilde{G}(-\omega)$ and $f(-\omega) = f(\omega)$, we can derive for Eq. (E.10) the expression

$$D_j(I_j) = \bar{D}_j \langle I \rangle I_j$$

[2]The distribution function $f(\omega)$ is defined by Eq. (D.1) on page 53.

with

$$\bar{D}_j = \frac{\kappa^2\omega_j^2}{Q^2} \sum_{(m)} \sum_{(l)} \int_{-\infty}^{\infty} d\omega \; f(\omega) \; \left|\widetilde{G}[(l+Q)\omega]\right|^2 \frac{\sin^2[(m+Q)\omega_j - (l+Q)\omega]\Delta T/2}{[(m+Q)\omega_j - (l+Q)\omega]^2 \Delta T/4}. \tag{E.11}$$

We next consider the integral over the frequency ω in detail. Generally,

$$\frac{\sin^2(\omega-\omega_0)\Delta T}{(\omega-\omega_0)^2\Delta T} \longrightarrow \pi\delta(\omega-\omega_0) \qquad \text{for} \qquad \Delta T \longrightarrow \infty$$

so that for sufficiently large ΔT and smooth functions $g(\omega)$ we can write

$$g(\omega)\,\frac{\sin^2(\omega-\omega_0)\Delta T}{(\omega-\omega_0)^2\Delta T} \approx g(\omega_0)\,\frac{\sin^2(\omega-\omega_0)\Delta T}{(\omega-\omega_0)^2\Delta T}.$$

which allows in Eq. (E.11) the substitutions

$$\left|\widetilde{G}[(l+Q)\omega]\right|^2 \longrightarrow \left|\widetilde{G}[(m+Q)\omega_j]\right|^2 \qquad \text{and} \qquad f(\omega) \longrightarrow f\!\left(\frac{m+Q}{l+Q}\,\omega_j\right).$$

The remaining integral can be evaluated in a closed form, yielding [16]

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega \, \frac{\sin^2[(m+Q)\omega_j - (l+Q)\omega]\Delta T/2}{[(m+Q)\omega_j - (l+Q)\omega]^2\Delta T/4} &= \frac{8}{(l+Q)^2\Delta T} \int_0^{\infty} d\omega \, \frac{\sin^2(l+Q)\omega\Delta T/2}{\omega^2} \\
= \frac{8}{(l+Q)^2\Delta T}\,\frac{\pi}{2}\,\frac{|l+Q|\Delta T}{2} &= \frac{2\pi}{|l+Q|}.
\end{aligned}$$

With that, Eq. (E.11) finally becomes

$$\bar{D}_j = \frac{2\pi\kappa^2\omega_j^2 N}{Q^2} \sum_{(m)} \left|\widetilde{G}[(m+Q)\omega_j]\right|^2 \sum_{(l)} \frac{1}{|l+Q|}\, f\!\left(\frac{m+Q}{l+Q}\,\omega_j\right). \tag{E.12}$$

This expression allows for overlapping frequency bands in the beam spectrum. In the special case that the bands are still separated, Eq. (E.12) reduces because then the summations will only contribute for $l = m$, resulting in

$$\bar{D}_j = \frac{2\pi\kappa^2\omega_j^2 N}{Q^2}\, f(\omega_j) \sum_{(m)} \frac{\left|\widetilde{G}[(m+Q)\omega_j]\right|^2}{|m+Q|}.$$

Appendix F

Parameters of the Cooling System

F.1 The Calculation of the Cooling Rate

For linear transverse stochastic cooling, we can deduce an expression for the cooling rate τ_{cool}^{-1} by integrating the Fokker-Planck equation. Here, the cooling rate is defined by

$$\tau_{cool}^{-1} = \frac{1}{\langle I \rangle} \frac{d}{dt} \langle I \rangle.$$

Assuming a distribution function $\rho(I,t)$ normalized to unity which vanishes beyond a maximum value I_{max}, we stipulate the boundary conditions

$$\int_0^\infty dI\ \rho(I,t) = 1 \qquad \text{and} \qquad \rho(I,t) \equiv 0 \quad \text{for} \quad I > I_{max}.$$

Starting with the Fokker-Planck equation (4.7),

$$\frac{\partial}{\partial t}\rho(I,t) = -\frac{\partial}{\partial I}\left\{ \bar{F} I \rho(I,t) - \frac{1}{2}\bar{D}\langle I \rangle I \frac{\partial}{\partial I}\rho(I,t) \right\},$$

we operate with $\int dI\, I$ on both sides of this equation, and obtain

$$\frac{d}{dt}\langle I \rangle \ = \ \int_0^\infty dI\ I \frac{\partial}{\partial t}\rho(I,t) \ = \ -\int_0^\infty dI\ I \frac{\partial}{\partial I}\left\{ \bar{F} I \rho(I,t) - \frac{1}{2}\bar{D}\langle I \rangle I \frac{\partial}{\partial I}\rho(I,t) \right\}.$$

Taking into account the boundary conditions, subsequent partial integrations yield

$$\frac{d}{dt}\langle I \rangle \ = \ \bar{F}\int_0^\infty dI\ I\rho(I,t) + \frac{1}{2}\bar{D}\langle I \rangle \int_0^\infty dI\ \rho(I,t) \ = \ \left\{ \bar{F} + \frac{1}{2}\bar{D} \right\} \langle I \rangle.$$

Hence the cooling rate follows as

$$\tau_{cool}^{-1} = \bar{F} + \frac{1}{2}\bar{D}. \tag{F.1}$$

F.2 The Derivation of the Optimum Gain

In this section, we will calculate the optimum gain which provides the most effective cooling operation and thus results in the maximum cooling rate. To that end, we investigate the gain $\widetilde{G}(\Omega)$ at a given frequency $\Omega_n = (n+Q)\omega'$ and henceforth consider the cooling rate as a function of the amplitude $G_n = |\widetilde{G}(\Omega_n)|$. The coefficients (E.6) and (E.12) entering into the cooling rate (F.1) therefore contain only the contribution from the frequency band investigated, i.e. $m = n$. To obtain an upper limit for cooling rate, we further assume that all particles experience the maximum cooling interaction, permitting us to set $\sin \Phi_n = 1$ in Eq. (E.6). Then the coefficients can be written as

$$\bar{F_n} = -\frac{\kappa\omega'}{Q}\,|\widetilde{G}(\Omega_n)|$$

and

$$\bar{D_n} = \frac{2\pi\kappa^2\omega'^2 N}{Q^2}\,\left|\widetilde{G}(\Omega_n)\right|^2 \sum_{(l)} \frac{1}{|l+Q|}\, f\!\left(\frac{\Omega_n}{l+Q}\right).$$

The amplitude $|\widetilde{G}_{opt}(\Omega_n)|$ of the optimum gain resulting in the maximum cooling rate τ_n^{-1} at the frequency Ω_n can be inferred from

$$\frac{d\tau_n^{-1}}{dG_n} = \frac{d}{dG_n}\left(\bar{F_n} + \frac{1}{2}\bar{D_n}\right) = 0 \qquad \text{for} \qquad G_n = |\widetilde{G}_{opt}(\Omega_n)|.$$

The elementary calculation yields

$$\frac{1}{|\widetilde{G}_{opt}(\Omega_n)|} = \frac{2\pi\kappa\omega' N}{Q} \sum_{(l)} \frac{1}{|l+Q|}\, f\!\left(\frac{\Omega_n}{l+Q}\right). \tag{F.2}$$

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